Dynamic Elections and Ideological Polarization[⋆]

Salvatore Nunnari[*]
salvatore.nunnari@unibocconi.it

Jan Zápal[**‡]
j.zapal@cerge-ei.cz

June 22, 2017

Abstract

How does political polarization affect the welfare of the electorate? We analyze this question using a framework in which two policy and office motivated parties compete in an infinite sequence of elections. We propose two novel measures to describe the degree of conflict among agents: *antagonism* is the disagreement between parties; *extremism* is the disagreement between each party and the representative voter. These two measures do not coincide when parties care about multiple issues. We show that forward-looking parties have an incentive to implement policies favored by the representative voter, in an attempt to constrain future challengers. This incentive grows as antagonism increases. On the other hand, extremism decreases the electorate's welfare. We discuss the methodological and empirical implications for the existing measures of political actors' ideal points and for the debate on elite polarization.

JEL Classification: C72, C73, D72, D78
Keywords: Political Polarization; Dynamic Elections; Multiple Issues; Electorate's Welfare.

[⋆] A previous version of this paper circulated under the title "Dynamic Policy Competition, Ideological Polarization and the Value of Veto Rights." We are grateful to Enriqueta Aragones, Torun Dewan, Daniel Diermeier, John Duggan, Jon Eguia, Tim Feddersen, Justin Grimmer, Massimo Morelli, Pablo Montagnes, Clara Ponsati, Juan Ortner, Howard Rosenthal, and seminar participants at IAE-CSIC, the Conference in Honor of Norman Schofield at Washington University in St Louis, the 2013 EPSA Conference in Barcelona, the CRENoS IV Workshop on Institutions, Individual Behavior and Economic Outcomes in Alghero, the 2013 APSA Annual Meeting in Chicago, the 2014 SAET Conference on Current Trends in Economics in Tokyo, Northwestern University, the University of Chicago, Princeton University, and the University of Mannheim for helpful comments and interesting discussions. The suggestions by the editor and three anonymous referees are also gratefully acknowledged. Jan Zapal acknowledges financial support from the Post-Doc Research Fund of the Charles University in Prague. All remaining errors are our own.

[*] Bocconi University and IGIER, Via Roentgen 1, Milano, 20136, Italy

[**] CERGE-EI, a joint workplace of Charles University and the Economics Institute of the Czech Academy of Sciences, Politickych veznu 7, 111 21 Prague, Czech Republic

[‡] IAE-CSIC and Barcelona GSE, Campus UAB, Bellaterra, Barcelona, 08193, Spain

1 Introduction

A large body of recent research in political science has been devoted to identifying and explaining ideological polarization, especially but not only in the United States. There is strong evidence that the U.S. Congress has grown progressively more polarized since the 1970s (McCarty, Poole, and Rosenthal, 2006), as well as some evidence of a polarization trend in presidential platforms (Budge, Klingemann, Volkens, Bara, and Tanenbaum, 2001; Klingemann, Volkens, Bara, Budge, and McDonald, 2006). Many empirical studies link this increasing ideological divide between the main American parties to legislative gridlock, elite incivility, income inequality, and voter disengagement (Layman and Carsey, 2002; Fiorina and Abrams, 2008; Hetherington, 2001). Across a broader range of countries, polarization is associated with democratic breakdown, corruption, and economic decline (Brown, Touchton, and Whitford, 2011; Frye, 2002; Linz and Stepan, 1978). To complement these empirical findings, many existing models of electoral competition and policymaking show that parties with conflictual preferences are socially suboptimal (Persson and Svensson, 1989; Alesina and Tabellini, 1990; Azzimonti, 2011). However, as we hope to show, these conclusions depend crucially on a number of stylized assumptions about the nature of the disagreement among political actors. While we lack a precise notion of ideological polarization when parties care about multiple dimensions, how we measure this concept crucially affects the conclusions on its welfare consequences.

In this article, we develop a model designed to study the policy consequences of ideological conflict when political actors have preferences over public policies on two separate issues.[1] We highlight that—when parties and voters care about multiple issues—we can use many different measures to describe the degree of conflict—or polarization—among agents' preferences in the political arena, and we propose two such measures. The first measure, which we label *extremism*, is the ideological distance of each party from the decisive voter in the electorate. The second measure, which we call *antagonism*, is the ideological distance that separates the two parties from each other and summarizes the degree of political competition between poli-

[1] In the U.S. and Europe, voters' opinions and political representatives' voting behavior can be organized and explained by two dimensions. The first dimension can be interpreted as government intervention in the economy. The second dimension picks up social issues or regional conflicts. Some scholars have argued that, since 1980, the political landscape in the U.S. is well summarized by a single dimension (Poole and Rosenthal, 1997; Shor and McCarty, 2011, but see Aldrich, Rohde, and Tofias, 2007 for a contrasting opinion). Nonetheless, both dimensions are valuable for analyzing coalition formation and political competition in other historical periods and countries (Poole and Rosenthal, 1997; Kollman, Miller, and Page, 1998; Miller and Schofield, 2003, 2008; Schofield, Miller, and Martin, 2003; Albright, 2010; Stoll, 2010).

cymakers. These two measures coincide in a one dimensional policy space, where the ideological distance between the two parties can increase only as they move further away from the representative voter.[2] However, they do not coincide in a two-dimensional setting: here, the two parties can be very close (when they share views on both dimensions) or very different (when they are perfectly opposed in one dimension), without altering their overall distance from the representative voter.

In our model, parties run in the many elections of an infinite horizon and commit to enact the same policy for their entire tenure in office. This *incumbent policy persistence* reflects politicians' inability to credibly promise policies different than those implemented while in office. This constraint is well documented and arises for many concurring reasons: internal party politics which generates organizational hysteresis (Miller and Schofield, 2003); voters' focus on parties' records and neglect of novel campaign promises (*retrospective voting*, Fiorina, 1981); the electoral costs of changing policy position and being perceived as *flip-floppers* (Adams, Clark, Ezrow, and Glasgow, 2006; DeBacker, 2015; Tavits, 2007; Tomz and Van Houweling, 2008, 2012). Not all parties face the same constraints, though. Following an electoral defeat, parties usually replace their leaders. This, together with the fact that their past policy platform has not been enacted and observed by voters, means that challengers are better able to credibly change their policy stance.[3] We assume that incumbents who remain in office find it too costly to implement policies that differ from those of their previous term. In particular, we model two parties who compete in an election in each of an infinite number of periods; the opposition party proposes a policy platform while the incumbent party, if reelected, enacts the same policy from the previous period. A representative voter picks her favorite candidate. The election winner implements the proposed policy and commits to do so for the duration of its tenure. In this sense, the identity of the incumbent party and its policy represent a dynamic linkage across periods.

There are two main questions that we wish to address with this simple setting. First, what is the impact of ideological disagreement on implemented policies when parties are long-lived and care about present as well as future electoral outcomes? Second, how do electoral competition, ideological disagreement and parties' dynamic incentives affect the electorate's welfare?

[2] In the classic framework where the two parties' ideal policies are on opposite sides of the ideal policy of the representative voter.

[3] Janda, Harmel, Edens, and Goff (1995) and Somer-Topcu (2009) analyze electoral manifestos data and show that this is indeed the case: parties that lost votes in previous elections change their programs more than parties what won votes.

We fully characterize a Stationary Markov Perfect Equilibrium (SMPE) of the dynamic electoral competition described above and prove it exists for any discount factor, any initial incumbent's policy, and any degree of antagonism and extremism.[4] According to the results of our model, parties alternate in power and long run policies tends to be more moderate, in the sense of being closer to the preferences of the representative voter: (i) the larger is the degree of antagonism between alternating governments; (ii) the smaller is parties' extremism; (iii) the larger is parties' patience. Opposition parties' ability to design a winning policy around the incumbents' commitment drives alternation in power. The key idea behind moderation is that an opposition party—which knows it is tying its hands with the current policy platform and, hence, will be vulnerable to future electoral defeat—has incentives to behave strategically by offering policies that restrict the set of policies future challengers can win an election with. In particular, the opposition party restricts the future challenger by contesting the election with a platform closer to the median voter's ideal policy: winning with a more moderate platform today makes the median voter more demanding in the following election. The more the preferences of the challenger depart from the preferences of the current government, that is, the higher antagonism is, the more the challenger will try to restrict the future opposition's ability to win an election with a platform close to its ideal policy. On the other hand, the degree of conflict between the parties and the representative voter, that is, extremism, does not affect the strategic incentives to moderate: increasing extremism increases both the marginal benefit of moderation (that is, the gain from constraining the future incumbent to more moderate policies) and its marginal cost (that is, the loss from implementing policies further away from the bliss point), leaving their ratio constant. As a consequence, a more extreme party will generally offer a more extreme policy platform.

This analysis suggests that the influence voters exert on policies is a function of the degree of antagonism and extremism of the political system. In particular, polarization can have counter-intuitive welfare implications: the electorate is best served by highly antagonist political elites that are perfectly opposed on one dimension.

[4] The only general existence result for dynamic elections applies to settings with countable state spaces (Duggan and Forand, 2013). In our model the state space is uncountable and, thus, proving existence is a necessary step of the analysis. Moreover, if we were to consider a model with a countable set of policies, the results in Duggan and Forand (2013) would guarantee existence of an equilibrium but would not provide a characterization of its dynamics or comparative statics with respect to patience, antagonism and extremism.

2 Related Literature

This paper contributes primarily to the theoretical literature on the relationship between political elites' polarization and policy outcomes. In a multidimensional and static framework, Krasa and Polborn (2014) study how ideological polarization on one dimension influences the candidates' positions on a second dimension. In most dynamic models, conflictual political preferences are socially suboptimal (Persson and Svensson, 1989; Alesina and Tabellini, 1990; Azzimonti, 2011; Prato, 2016). Our paper shares with this literature the idea that forward-looking incumbents have incentives to strategically position current policies to affect future political outcomes, and that these incentives are stronger when the conflict of preferences is starker. On the other hand, in our model disagreement can be over multiple dimensions and the channel to constrain a future incumbent is the demands of the electorate rather than inefficient or misdirected spending. Our novel approach shows that—when the ideological conflict is multidimensional—preference divergence does not necessarily lead to higher inefficiencies and welfare losses for the electorate, but it could have the opposite effect.

More recent studies argued that polarized parties and divergent platforms can be welfare enhancing (Bernhardt, Duggan, and Squintani, 2009; Bernhardt, Campuzano, Squintani, and Camara, 2009; Van Weelden, 2013, 2015). In these works, polarization coincides with our notion of extremism and helps the electorate through channels different than the one highlighted in our paper. Van Weelden (2013, 2015) shows that, in a more polarized political environment, the incumbent gives up rent extraction for fear of being replaced by a challenger with markedly different policy preferences. Bernhardt et al. (2009) analyze a static and unidimensional electoral competition where more polarized parties propose more extreme electoral platforms; for moderate levels of polarization, this is welfare enhancing because it provides the median voter with a more varied choice. Bernhardt et al. (2009) study repeated elections where candidates' types are private information and show that the incumbent will compromise more if his potential replacement is drawn from the other side of the political spectrum. In our case, on the other hand, the beneficial effect of antagonism comes from more moderate policies implemented strategically by forward-looking incumbents of known type, and increased extremism is always detrimental to the representative voter.

Our work is related to a growing theoretical literature on dynamic elections with endogenous economic or political state variables (Krusell and Rios-Rull, 1999; Bai and Lagunoff, 2011; Battaglini, 2014). The key assumption in our model, namely, the differential ability of incumbents and challengers to adjust policy platforms, owes to the seminal work of Kramer (1977) and Wittman (1977), who pioneered the study of dynamic models of asynchronous policy competition. While Kramer (1977) and Wittman (1977) focus on myopic parties, Forand (2014) considers farsighted parties who take into account future opponents' policy choices and offer policy platforms on a single dimension. With a single dimension, the ideological distance between the two parties coincide with the disagreement between the parties and the median voter. In this framework, the incentive to moderate in order to constrain future incumbents exists but it is unchanged as we increase the ideological distance between the two parties. To the contrary, the parties and voters in our setup care about multiple dimensions. Expanding the policy space beyond a single dimension allows us to explore the different facets of ideological polarization and to highlight the ambiguous impact of parties' conflict of preferences on the electorates' welfare.[5]

Less closely related to this paper is the literature on dynamic elections with adverse selection and/or moral hazard (for example Duggan, 2000; Bernhardt, Dubey, and Hughson, 2004; Banks and Duggan, 2009; Kalandrakis, 2009, see Duggan and Martinelli, 2015 for comprehensive survey). This literature stresses the link between parties' policies and their re-election prospects and, typically, features some degree of incumbency advantage due to the electorate's greater information on the incumbent. In our model, instead, the incumbent's commitment to a policy and the lack of any uncertainty generate deterministic alternation in power. Therefore, the policy platform chosen by the challenger in a given election only affects the policy with which her opponent will defeat her in the next election.

Finally, our paper is also formally related to models of dynamic legislative bargaining with an endogenous status quo and farsighted players (Baron, 1996; Kalandrakis, 2004; Diermeier and Fong, 2011; Bowen, Chen, and Eraslan, 2014; Dziuda and Loeper, 2016). To view our model as a legislative bargaining model, reinterpret the representative voter as the median legislator, the parties as the only legislators that have the power to set the agenda[6], and define the state variable as the status quo policy, that is, the policy implemented by the legislature in the previous period.

[5] Nonetheless, our results are related to Forand's. Reducing our model to one dimension, the long-run policies from Proposition 1 coincide with the long-run bound on extremism characterized in Forand (2014).

[6] These two legislators can be interpreted as the median legislators or party leaders of the two parties, as in the procedural cartel theory introduced by Cox and McCubbins (1993, 2005).

3 Model and Equilibrium Notion

We describe here a dynamic model of electoral competition between policy and office motivated parties. In each period of an infinite horizon, two parties, 1 and 2, compete in an election decided by a representative voter v. Each period starts with a party as the incumbent office holder and the other party, the opposition, striving to replace it in power. A key feature of our model is *incumbent policy persistence*: incumbents cannot distance themselves from their past policies, while opposition parties can put forward new policies more freely.

The Electoral Process. Elections occur at the beginning of each period. The opposition party contests an election by offering a bi-dimensional policy $\mathbf{p} = (p^1, p^2) \in X = \mathbb{R}^2$, or stays out of the race. The incumbent party is committed to the policy $\mathbf{q} \in X$ that brought it into office. The representative voter[7] is, thus, confronted with the choice between $\mathbf{p}$, promised by the opposition party, and $\mathbf{q}$, the continuing policy of the incumbent party. The elected party implements its winning policy and becomes the incumbent at the beginning of the next period. The policy implemented in a period becomes the incumbent's policy commitment in the next period and, as such, represents a dynamic linkage between periods.

Stage Utilities. The stage utility player $i \in \{1, 2, v\}$ receives from policy $\mathbf{p} \in X$ is measured by the squared distance of $\mathbf{p}$ from i's *bliss point*, or ideal policy, $\mathbf{b}_i = (b_i^1, b_i^2)$:

$$u_i(\mathbf{p}) = -(p^1 - b_i^1)^2 - (p^2 - b_i^2)^2 \tag{1}$$

Denoting by $d(\mathbf{x}, \mathbf{y})$ the usual Euclidean distance between $\mathbf{x} \in X$ and $\mathbf{y} \in X$, we can rewrite equation (1) as $u_i(\mathbf{p}) = -d^2(\mathbf{p}, \mathbf{b}_i)$.[8] We abuse notation slightly and denote by $d(\mathbf{x}) = ||\mathbf{x}||$ the distance from the origin of $\mathbf{x} \in X$. In addition to policy, parties care about being in office. The party in power in a period receives office rents $r \geq \bar{r}$, where $\bar{r}$ is determined and its role discussed in the equilibrium analysis below.

[7] A necessary and sufficient condition for the existence of a decisive median voter in a multidimensional policy space is the 'radial symmetry' of voters' ideal policies (see Plott, 1967 and Duggan, 2012). Radial symmetry obtains, for example, when voters' ideal policies are distributed according to a radially symmetric density, such as a bi-variate normal or uniform distribution on a disk in $\mathbb{R}^2$ (as in Baron, Diermeier, and Fong, 2012). Stage-game median is also dynamic-game median under quadratic stage utilities (Banks and Duggan, 2006).

[8] We use quadratic Euclidean preferences primarily for convenience. In Appendix A2.1 we present a model with general utility functions that are continuous, decreasing and weakly concave in $d(\mathbf{x}, \mathbf{y})$, and prove results analogous to the ones presented below.

Office-holding benefits include patronage positions in government and government-owned companies, public financing of party activities and other office perks that are consumed only by the party in government. The utility i derives from a sequence of policies $\mathbf{P} = \{\mathbf{p}_0, \mathbf{p}_1, \ldots\}$ is the discounted sum of payoffs from each period:

$$U_i(\mathbf{P}) = \sum_{t=0}^{\infty} \delta_i^t \left[u_i(\mathbf{p}_t) + \mathbb{I}_{i,t} \cdot r\right] \tag{2}$$

where $\delta_i \in [0,1)$ is player i's discount factor and $\mathbb{I}_{i,t} = 1$ when party i is in power in period t and zero otherwise. The representative voter receives no utility from r, hence $\mathbb{I}_{v,t} = 0$ for any t. We assume $\delta_1 = \delta_2 \equiv \delta$ and $\delta_v = 0$.[9]

Parties' Antagonism and Extremism. The model is shift and rotation invariant, so, without loss of generality, we define the representative voter's bliss point as the origin of the plane, $\mathbf{b}_v = (0,0)$, party 1's bliss point as $\mathbf{b}_1 = (b_1, 0)$ and party 2's bliss point as $\mathbf{b}_2 = (b_2^1, b_2^2)$ with $d(\mathbf{b}_2) = b_2$. The distance of the bliss point of party $i \in \{1,2\}$ from the bliss point of the representative voter is b_i. The parameter b_i, thus, captures the ideological distance between the voter and the party. We call this the degree of *extremism* of party i.

A second, separate, measure of ideological divergence is given by the angle $\alpha \in [0,\pi]$ formed by the two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.[10] This parameter captures how different the parties' bliss points are from each other, regardless of their distance from the representative voter's ideal policy. When $\alpha = 0$, the parties' bliss points are on the same ray departing from the origin. When $\alpha \in (0,\pi)$, the two parties diverge on both dimensions, keeping the distance from the origin of the plane constant. When $\alpha = \pi$, the two parties are perfectly opposed on one dimension and share the same ideology on the second dimension. We call α the degree of *antagonism* of the parties.

While we define antagonism and extremism using a two-dimensional policy-space, X, our definitions and results would be identical in a similar model with more than two dimensions. In particular, the equilibrium characterized in Proposition 1 survives in a model with $X = \mathbb{R}^n$. In this model, the definitions of antagonism and extremism, as well as the equilibrium strategies, will take place on the two-

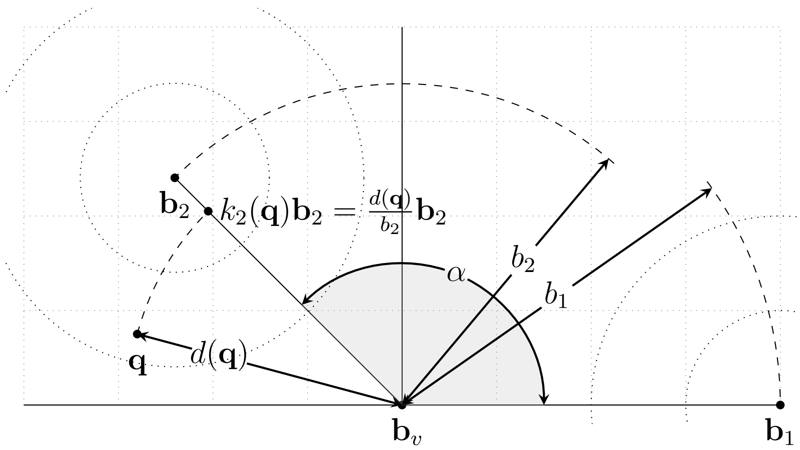
[9] Assuming that the representative voter is myopic facilitates the exposition but it is not needed for our results. In Appendix A2.2 we present a model with forward-looking v and show that, for any equilibrium discussed in the main text, there exists an equilibrium with forward-looking v with identical comparative static on the representative voter's welfare with respect to parties' antagonism and extremism.

[10] Notice that $\cos\alpha = \frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{b_1 b_2}$ where $\mathbf{b}_1 \cdot \mathbf{b}_2$ is the usual inner product.

dimensional hyperplane in $\mathbb{R}^n$ that contains the bliss points of the parties and of the representative voter.

The equilibrium strategies we characterize below depend solely on the distance between the incumbent's policy and the bliss point of the representative voter, that is, the origin of the plane. We denote with $k_i(\mathbf{x}) = \frac{d(\mathbf{x})}{b_i} \geq 0$ for $i \in \{1, 2\}$ the distance of policy $\mathbf{x} \in X$ from the origin, relative to b_i. With this notation, $k_i(\mathbf{q})\mathbf{b}_i$ is a point on the line connecting $\mathbf{b}_v$ with $\mathbf{b}_i$, at the same distance from $\mathbf{b}_v$ as the incumbent's policy commitment $\mathbf{q}$. Figure 1 shows the basic model parameters: a set of arbitrary bliss points for the three players ($\mathbf{b}_v$, $\mathbf{b}_1$, and $\mathbf{b}_2$) and the indifference curves generated by their Euclidean preferences over policies; the corresponding degree of antagonism (α) and extremism (b_1, b_2); the incumbent's policy commitment ($\mathbf{q}$) and its distance from the origin ($d(\mathbf{q})$); and a point on the line connecting $\mathbf{b}_2$ with the origin, at the same distance from the origin as the incumbent's policy ($k_2(\mathbf{q})\mathbf{b}_2$). Figure 2 shows examples of parties' bliss points for two different degrees of antagonism ($\alpha' > \alpha$) and two different degrees of extremism ($b_i' > b_i$).

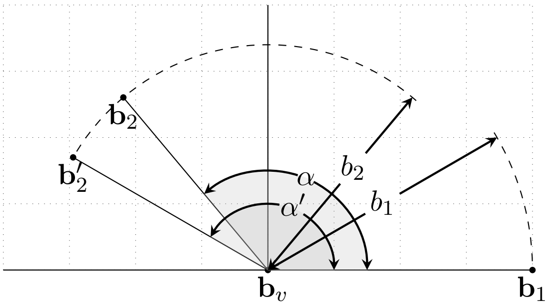
Figure 1: Basic Model Parameters



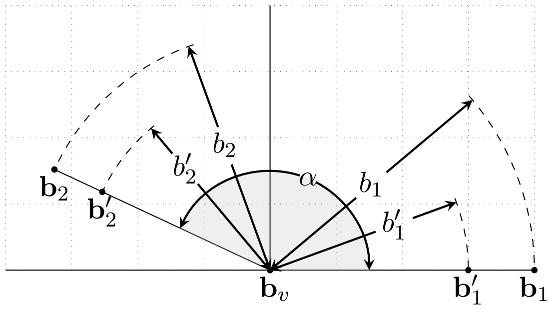
Strategies. We focus on equilibria in pure Markov strategies (Maskin and Tirole, 2001). We assume that the decision of the opposition party regarding which policy to run with, should it contest the election, depends solely on the incumbent's identity and the policy it is committed to. Markovian strategies that abstract from the history of play are standard in dynamic models of political economy (Baron, 1996; Kalandrakis, 2004, 2010; Battaglini, Nunnari, and Palfrey, 2012; Duggan and Kalandrakis, 2012; Duggan and Forand, 2013; Forand, 2014), capture the simplest form of behavior consistent with rationality, and clearly isolate the underlying strategic

Figure 2: Parties Bliss Points for Different Levels of Antagonism and Extremism

(a) Antagonism (α)



(b) Extremism (b_1, b_2)



motives shaping the competition between the two parties in a dynamic environment, independent of the time horizon. Additionally, the two parties interact over a long time horizon and can be represented by different politicians in different points in time. Therefore, strategies that potentially depend on events from the (distant) past and require coordination might be excessively demanding and inappropriate for the context at hand. Given the policies of the two parties in a contested election, we assume the representative voter elects the party running with the policy she prefers and votes for the opposition when indifferent.[11]

Definition 1. *A Stationary Markov strategy for the opposition party $i \in \{1, 2\}$, given incumbent $j = \{1, 2\} \setminus \{i\}$, is a function $\sigma_i : \{j\} \times X \to X \cup \{Out\}$, mapping j's policy commitment $\mathbf{q} \in X$ into an electoral platform $\mathbf{p} \in X$ or the decision not to contest the election ('Out'). A Stationary Markov strategy for the representative voter v is a function $\sigma_v : (\{1, 2\} \times X) \times X \to \{Yes, No\}$ that maps j's policy commitment $\mathbf{q} \in X$ and the electoral platform of the opposition $\mathbf{p} \in X$ into the decision to elect the opposition.*

Dynamic Utilities. We denote by $V_i^j(\mathbf{q}|\sigma)$ the dynamic utility party $i \in \{1, 2\}$ derives from the infinite sequence of policies generated by the profile of strategies $\sigma = (\sigma_1, \sigma_2, \sigma_v)$, at the beginning of a period with incumbent party $j \in \{1, 2\}$ committed to $\mathbf{q}$. Formally, if $\mathbf{P}(j, \sigma, \mathbf{q}) = \{\mathbf{p}_0, \mathbf{p}_1, \ldots\}$ is a path of policies generated

[11] This is a standard assumption in models involving voting over endogenous (proposed) alternatives (see Baron, 1996; Diermeier and Fong, 2011; Duggan and Kalandrakis, 2012; Bowen et al., 2014; Forand, 2014, among others) as it guarantees that the set of policies the representative voter accepts is closed. When the indifferent voter always votes for the incumbent, an equilibrium fails to exist. While there is a well documented incumbency advantage in American politics, this slight incumbency disadvantage suits well governments in parliamentary systems (Rose and Mackie, 1983; Strom, 1985; Powell and Whitten, 1993; Veiga and Veiga, 2004) or legislators in developing countries or weak party systems (see Uppal, 2009; Klašnja and Titiunik, 2017; Klašnja, 2016).

by play according to σ, starting from an incumbent j committed to $\mathbf{q}$, we have:

$$V_i^j(\mathbf{q}|\sigma) = U_i(\mathbf{P}(j,\sigma,\mathbf{q})) = \sum_{t=0}^{\infty} \delta^t \left[u_i(\mathbf{p}_t) + \mathbb{I}_{i,t} \cdot r\right] \tag{3}$$

Equilibrium Notion. We look for a stationary Markov perfect equilibrium.

Definition 2. *A Stationary Markov Perfect Equilibrium (SMPE) is a profile of Stationary Markov strategies $\sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_v^*)$ such that, for any $\mathbf{q} \in X$,*

$$\sigma_i^* \in \arg\max_{\sigma_i} V_i^j(\mathbf{q}|(\sigma_i, \sigma_j^*, \sigma_v^*)) \tag{4}$$

for any $i \in \{1,2\}$ and $j = \{1,2\} \setminus \{i\}$, and $\sigma_v^*(j,\mathbf{q},\mathbf{p}) =$ Yes if and only if

$$u_v(\mathbf{p}) \geq u_v(\mathbf{q}). \tag{5}$$

An equilibrium, as specified in (5), requires that the representative voter supports the opposition if and only if her expected utility from the incumbent's policy is not larger than the expected utility from the opposition's platform. The fact that the opposition optimizes its dynamic utility is ensured by (4).

4 Equilibrium Analysis

4.1 Simple Strategies

In the remainder, we focus on a class of SMPE of the dynamic electoral competition game, where the two parties use strategies of a simple form, captured by a single parameter $\hat{k}_i$.

Definition 3. *A Simple (Stationary Markov) strategy σ_i for $i \in \{1,2\}$, satisfies*

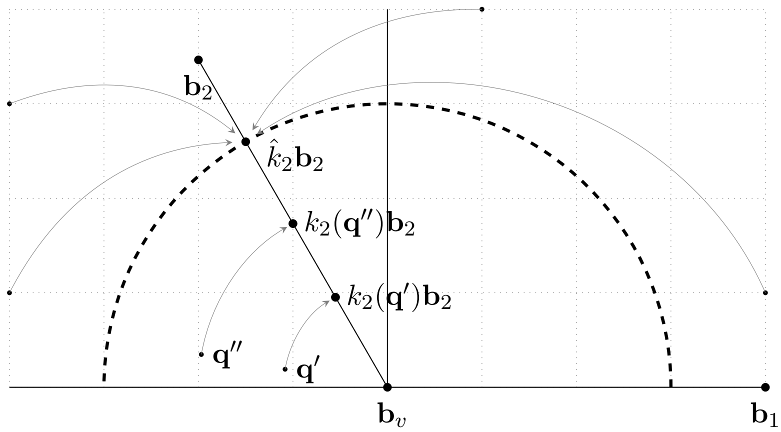
$$\sigma_i(\mathbf{q}) \equiv \mathbf{p}_i(\mathbf{q}) = \begin{cases} k_i(\mathbf{q})\mathbf{b}_i & \text{for } \; k_i(\mathbf{q}) \leq \hat{k}_i \\ \hat{k}_i \mathbf{b}_i & \text{for } \; k_i(\mathbf{q}) \geq \hat{k}_i \end{cases} \tag{6}$$

where $\hat{k}_i$ is a parameter. A Simple Stationary Markov Perfect Equilibrium (SSMPE) is a SMPE where the parties use simple strategies.[12]

[12] In Appendix A2.3 we show that the policy dynamics generated by the unique SSMPE char-

As discussed above, $k_i(\mathbf{q})$ measures the distance of the incumbent's policy commitment $\mathbf{q}$ from the origin of the plane. According to these simple strategies, when $\mathbf{q}$ is closer than $\hat{k}_i$ to the origin, the opposition party $i \in \{1, 2\}$ contests the election with $k_i(\mathbf{q})\mathbf{b}_i$. This policy is located on the ray that connects $\mathbf{b}_v$ with $\mathbf{b}_i$, and it is at the same distance from $\mathbf{b}_v$ as the incumbent's policy commitment $\mathbf{q}$. When, instead, $\mathbf{q}$ is further than $\hat{k}_i$ from the origin, the opposition party $i \in \{1, 2\}$ runs with $\hat{k}_i\mathbf{b}_i$. This policy is on the same ray that connects $\mathbf{b}_v$ with $\mathbf{b}_i$, but it is $\hat{k}_i$ distant from the origin. Figure 3 shows the policies corresponding to these simple strategies: for a number of arbitrary party 1's policy commitments, the arrow indicates the policy platform chosen by party 2. Consider how the policies evolve as $k_2(\mathbf{q}) = \frac{d(\mathbf{q})}{b_2}$ increases, that is, as party 1's policy commitment moves further away from the origin: for low values of $k_2(\mathbf{q})$, the policy $\mathbf{p}_2(\mathbf{q})$ increases linearly—that is, it gets further away from the origin—as $k_2(\mathbf{q})$ increases; once $k_2(\mathbf{q})$ reaches $\hat{k}_2$, $\mathbf{p}_2(\mathbf{q})$ stays constant and it is not affected by a further increase of $k_2(\mathbf{q})$.

Figure 3: Simple Stationary Markov Proposal Strategy



4.2 Results

Proposition 1 below shows that a SSMPE of the dynamic electoral competition game exists and is generally unique, and it fully characterizes it.

Proposition 1 (SSMPE of Dynamic Electoral Competition Game)**.** *Assume, without loss of generality, that $b_1 \geq b_2$. Then:*

1. $\hat{k}_1 = 1$ *and* $\hat{k}_2 = \frac{1+\delta\cos\alpha}{1+\delta}$ *characterize a SSMPE;*

acterized below are identical to the ones generated by the unique subgame perfect equilibrium of the finite horizon version of our model.

2. *if* $b_1 > b_2$*, this SSMPE is unique; if* $b_1 = b_2$*, there exists exactly one additional 'mirror' SSMPE where* $\hat{k}_1$ *and* $\hat{k}_2$ *are reversed;*

3. *in any SSMPE, elections are contested and incumbents always defeated;*

4. *starting from* $\mathbf{q}^0$*, SSMPE policies converge to alternation between*

   (a) $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$ *and* $\hat{k}_2\mathbf{b}_2$*, if* $k_2(\mathbf{q}^0) \geq \hat{k}_2$

   (b) $k_1(\mathbf{q}^0)\mathbf{b}_1$ *and* $k_2(\mathbf{q}^0)\mathbf{b}_2$*, if* $k_2(\mathbf{q}^0) \leq \hat{k}_2$*.*

Proof. See Appendix A1.

In a SSMPE, the policy platforms chosen by party $i \in \{1, 2\}$ always lie on the ray starting at $\mathbf{b}_v$, the origin of the plane, and passing through $\mathbf{b}_i$, its ideal policy. We call such a ray a $\mathbf{b}_i$-ray. To understand why, consider the dynamic utility i derives from running with a policy $\mathbf{p}$ the representative voter prefers to $\mathbf{q}$: $u_i(\mathbf{p}) + r + \delta V_i^i(\mathbf{p}|\sigma)$. In this expression, the first two terms capture the current utility following an electoral victory. The third term captures the future stream of payoffs, given the policy commitment. This discounted value depends on $d(\mathbf{p})$ but not on the specific location of $\mathbf{p}$: the strategies of all players depend on the distance of the incumbent's policy commitment from the origin, but not on its exact location. As a result, when moving $\mathbf{p}$ along any circle centered at $\mathbf{b}_v$, the dynamic utility of i increases as $\mathbf{p}$ approaches the $\mathbf{b}_i$-ray. This increases the utility accrued in the current period but maintains constant the future utility.

Figure 4: Strategic Incentives in SSMPE

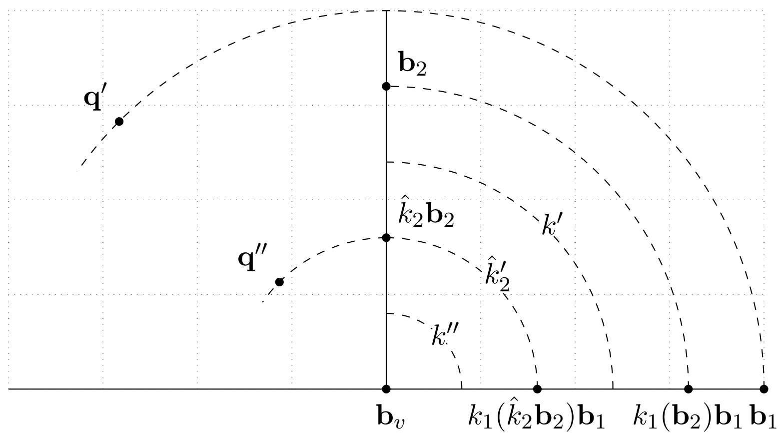


Figure 4 shows many different circles centered at $\mathbf{b}_v$, in dashed lines. These circles represent the indifference curves of the representative voter. For a given incumbent's policy commitment, $\mathbf{q}$, the representative voter elects the challenger if it proposes any policy at least as close to her bliss point, that is, on the boundary or strictly inside the corresponding indifference curve. For the parties, the simple strategies from Proposition 1 prescribe that party 2 runs with $\hat{k}_2\mathbf{b}_2$ for any incumbent's policy commitment $\mathbf{q}$ with $d(\mathbf{q}) \geq d(\mathbf{q}'')$ and runs with $k_2(\mathbf{q})\mathbf{b}_2$ for any $\mathbf{q}$ with $d(\mathbf{q}) \leq d(\mathbf{q}'')$. Similarly, party 1 runs with $\hat{k}_1\mathbf{b}_1 = \mathbf{b}_1$ or $k_1(\mathbf{q})\mathbf{b}_1$ depending on whether $d(\mathbf{q}) \gtreqless d(\mathbf{q}')$.

We first discuss the intuition underlying the strategy of party 2. When $d(\mathbf{q}) \leq d(\mathbf{q}'')$, running with $k_2(\mathbf{q})\mathbf{b}_2$ means running with a policy on the $\mathbf{b}_2$-ray with the same distance from the origin as $\mathbf{q}$. For incumbent's policy commitments close to the origin, party 2 is constrained by the demand of the representative voter and runs with a winning platform as close as possible to its bliss point $\mathbf{b}_2$. On the other hand, when $d(\mathbf{q}) \geq d(\mathbf{q}'')$, party 2 runs with $\hat{k}_2\mathbf{b}_2$, a policy strictly inside the voter's acceptance set. Consider, for example, an incumbent party 1 committed to $\mathbf{q}'$ from Figure 4. In this case, the voter is willing to elect a challenger party 2 that runs with its bliss point, $\mathbf{b}_2$, because this policy is closer to the voters's bliss point than $\mathbf{q}'$. A perfectly myopic opposition would propose this policy platform. However, a forward-looking opposition finds it optimal to run with policy $\hat{k}_2\mathbf{b}_2$, which is closer to $\mathbf{b}_v$ than $\mathbf{b}_2$. We call this behavior *moderation*.

The incentive to moderate is purely strategic and arises from the dynamic nature of the game. On one hand, moderating has a cost: running with $\hat{k}_2\mathbf{b}_2$, rather than $\mathbf{b}_2$, harms party 2 in the current period, since the policy it implements when in office is further away from its bliss point. On the other hand, moderating has a benefit: winning the current election with a more moderate policy helps in the following period; party 2 will be committed to a more moderate platform and this makes the representative voter more demanding. If party 1 wants to re-gain power, it is forced to propose $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$, a policy that both the representative voter and party 2 prefer to $k_1(\mathbf{b}_2)\mathbf{b}_1$ (what party 1 would propose with an incumbent committed to $\mathbf{b}_2$). The extent of moderation, $\hat{k}_2$, is then determined by the strength of two forces. The first force pushes the opposition's policy in the direction of its bliss point, in an attempt to increase its current utility. The second, strategic force, pushes the opposition's policy in the direction of the representative voters's bliss point, in an attempt to constrain the future behavior of the defeated incumbent. As we discuss in the following section, the magnitude of this latter force depends on the parties' patience and on the intensity of their ideological disagreement. The equilibrium extent of moderation is the value of $\hat{k}_2$ that balances the marginal static cost with the marginal dynamic benefit.

We now discuss the equilibrium strategy of party 1. Contrary to party 2, party 1 does *not* moderate: it proposes its ideal policy $\mathbf{b}_1$ whenever the representative voter prefers it to the incumbent's commitment. Why is this the case? While the two parties have similar strategic incentives to moderate, moderation is a *strategic substitute*. Assume party 1 moderates, that is, it proposes a policy closer to the origin than $\mathbf{b}_1$. This has a cost in the current period: all moderate policies reduce its current utility from holding office. On the other hand, since party 2 already moderates, not all moderate policies give a benefit in the following period, in terms of leading to future policies closer to its ideal point. Party 1 will affect the electoral strategy of its challenger only if it moderates to $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$ or to a policy even closer to the origin, for example, referring again to Figure 4, to $k''\mathbf{b}_1$. Moderating less—that is, running with $k'\mathbf{b}_1$ or any other policy between $\mathbf{b}_1$ and $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$—will not make any difference in the following election, as party 2 will run with $\hat{k}_2\mathbf{b}_2$ as prescribed by its moderating strategy. However, moderating to $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$ or more is too costly in terms of foregone current utility and party 1 abandons the idea of moderation altogether. This explains why an equilibrium in simple strategies must be asymmetric, with one party moderating and the other one sticking to its guns. The reason why party 2 is the one who moderates lies in the fact that party 1 has a weaker incentive to moderate: its ideal policy is more extreme and, thus, the future utility gain it gets from constraining its opponent is smaller.[13]

The assumed lower bound on office rents, $\bar{r} \equiv \max\{b_1, b_2\}^2(1+\delta)$, guarantees that, in equilibrium, both parties find it advantageous to contest elections rather than staying out of the race and leaving the incumbent indefinitely in power. To understand why, consider the case where party 1 is the incumbent and is committed to enact policy $\mathbf{q}$. Absent any office motivation, the opposition party can ensure a payoff of $\frac{u_2(\mathbf{q})}{1-\delta} \leq 0$ by never running for office. For party 2 to voluntarily contest an election for any state $\mathbf{q} \in X$, the optimal policy characterized in Proposition 1 must give her at least what she can get by running with $\mathbf{b}_v$, the bliss point of the representative voter. This policy platform ensures electoral victory for any $\mathbf{q}$ and gives party 2 a payoff of $\frac{u_2(\mathbf{b}_v)}{1-\delta} + \frac{r}{1-\delta^2}$. While sufficiently large office rents are necessary to construct an SSMPE, their magnitude has no effect on the equilibrium

[13] Notice that an equilibrium in simple strategies has to be asymmetric also when the degree of extremism of the two parties is the same, that is, when $b_1 = b_2$. In this case, the two parties have the same incentive to moderate, but moderation is still a strategic substitute. Provided one party moderates, the opponent has no incentive to moderate at all. As specified by Proposition 1, we have two asymmetric equilibria, one with party 1 moderating and another one with party 2 moderating. These equilibria are equivalent from the point of view of the representative voter.

policy platforms. This apparently surprising result is due to the lack of electoral uncertainty in our model: the opposition party which decides to contest an election either wins or loses for sure. In a similar model with a non-degenerate reelection probability, increasing office rents would weaken policy motivation; since a weaker policy motivation affects both the costs and benefits of moderation, increasing office rents would have an uncertain effect on the parties' incentive to moderate in equilibrium.

Long-Run Policies and Convergence Dynamics. Proposition 1 also specifies the long run policies that we converge to as a consequence of equilibrium strategies. We have three cases to consider. First, assume the initial incumbent's policy commitment, $\mathbf{q}^0$, is at least as close to the origin as the strategically induced bliss point of party 2, $\hat{k}_2\mathbf{b}_2$. In this case, the representative voter's acceptance set is binding in all elections and all policies will lie at the same distance from the origin as the initial incumbent's policy commitment. The policies implemented will alternate between $k_2(\mathbf{q}^0)\mathbf{b}_2$ and $k_1(\mathbf{q}^0)\mathbf{b}_1$, depending on the identity of the incumbent.

Second, assume the initial incumbent's policy commitment is further away from the origin than $\mathbf{b}_1$. If party 1 is at the opposition in the first period, it wins the election with its ideal policy $\mathbf{b}_1$. In the second period, party 2 wins the election with $\hat{k}_2\mathbf{b}_2$, a policy the representative voter strictly prefers to the incumbent's policy commitment $\mathbf{b}_1$. In all future periods, the policy implemented will be at the same distance from the origin, alternating between $\hat{k}_2\mathbf{b}_2$, when 2 is in power, and $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$, when 1 is in power. If party 2 is at the opposition in the first period, it wins the election with $\hat{k}_2\mathbf{b}_2$ and the policy dynamic immediately reaches alternation between $\hat{k}_2\mathbf{b}_2$ and $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$.

Finally, assume $\mathbf{q}^0$ is further away from the origin than $\hat{k}_2\mathbf{b}_2$ but closer to the origin than $\mathbf{b}_1$. This case is similar to the second one, except that, if party 1 is at the opposition in the initial period, it is constrained to offer $k_1(\mathbf{q}^0)\mathbf{b}_1$. From the second period, we have the same alternation between $\hat{k}_2\mathbf{b}_2$ and $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$.

5 Representative Voter's Welfare

The welfare of the representative voter from an ex-ante perspective, that is, at the beginning of the game, depends on the identity of the incumbent and on its policy commitment in the first period. Instead of making arbitrary assumptions, we assume that party 1 is the first-period incumbent with probability $\rho \in (0, 1)$ and that its policy commitment $\mathbf{q}^0$ is distributed according to a continuous cumulative distribution function $F(\mathbf{q}^0)$ with strictly positive density on X.

Definition 4. *In the SSMPE from Proposition 1, the ex-ante welfare of v is:*

$$W(\hat{k}_1, \hat{k}_2, b_1, b_2) = \int_X \rho u_v(\mathbf{p}_2(\mathbf{z})) + (1-\rho) u_v(\mathbf{p}_1(\mathbf{z}))\, d(F(\mathbf{z})). \tag{7}$$

Lemma 1 shows that the representative voter is strictly worse off as the degree of policy moderation observed in equilibrium decreases and as the parties bliss points diverge from the representative voter's bliss point.

Lemma 1. *In the SSMPE from Proposition 1, the ex-ante welfare of the representative voter is decreasing in $\hat{k}_2$, b_1, and b_2.*

Proof. See Appendix A1.

6 Comparative Statics

The following proposition formalizes the marginal impact of the model parameters on the strategic force pushing the parties towards moderation:

Proposition 2 (SSMPE Comparative Static)**.** *In the SSMPE from Proposition 1:*

$$\frac{\partial \hat{k}_2}{\partial \alpha} \leq 0 \qquad \frac{\partial \hat{k}_2}{\partial \delta} \leq 0 \tag{8}$$

Proof. Immediate.

Proposition 2 says that the strategic force pushing the more moderate party towards moderation gains strength as the ideological conflict between the parties becomes more pronounced (higher α) and as the future becomes more important (higher δ). As discussed in the previous section, parties have an incentive to moderate because they internalize that, if they locate closer to the median, then the rival will have to, as well. When parties are more patient, they care relatively more about what the rival will do next period, so the more moderate party will locate closer to the representative voter to induce the rival to do the same. The same occurs when ideological divergence rises: for the same distance from the representative voter, a party is hurt more by its rival's location when ideological divergence is greater.

Combining the comparative static results from Proposition 2 with Lemma 1, we have the following corollary about the marginal impact of antagonism, α, and extremism, b_i, on the representative voter's welfare.

Corollary 1 (SSMPE Representative Voter's Welfare Comparative Static)**.** *In the SSMPE from Proposition 1, the representative voter's ex-ante welfare, W,*

1. *is non-decreasing in α; increasing in α if $\delta > 0$;*
2. *is decreasing in b_i for $i \in \{1, 2\}$.*

Figure 5 shows how the two parties' equilibrium strategies change when we increase the value of α (Figure 5a) and b_2 (Figure 5b), in the limit as $\delta \to 1$. We focus here on the case where the initial incumbent's policy commitment is sufficiently far from the origin to generate interesting policy dynamics. When the initial incumbent's policy commitment is sufficiently close to the origin—that is, closer than $\hat{k}_2\mathbf{b}_2$—neither α nor b_2 matter for the moderation of implemented policies: in this case, neither party has an incentive to moderate further and the distance of implemented policies from the representative voter's ideal policy stays constant.[14]

We know from Proposition 1 and the discussion above that, when the initial incumbent's policy commitment is sufficiently far from the origin, the equilibrium policies converge, in at most two periods, to alternation between $\hat{k}_2\mathbf{b}_2$ and $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$. Figure 5 shows the policy implemented whenever party 2 is in power, $\hat{k}_2\mathbf{b}_2$.[15] While this neglects a component of the representative voter's ex-ante welfare—which also depends on the policy implemented by party 1—it captures the main intuition.[16] The dashed line in both figures traces $\hat{k}_2\mathbf{b}_2$ as α increases from 0 to π, and as b_2 increases from 0 to 4 (b_1 can be arbitrary on both figures provided $b_1 > b_2$).

When α increases, the extent of ideological conflict between the two parties increases. This strengthens the strategic force to moderate and, thus, reduces $\hat{k}_2$. Since the party's bliss point is unchanged but the weight applied to it has decreased, the equilibrium policy of party 2, $\hat{k}_2\mathbf{b}_2$, moves closer to the bliss point of the representative voter (Figure 5a). This clearly benefits the representative voter. Increasing

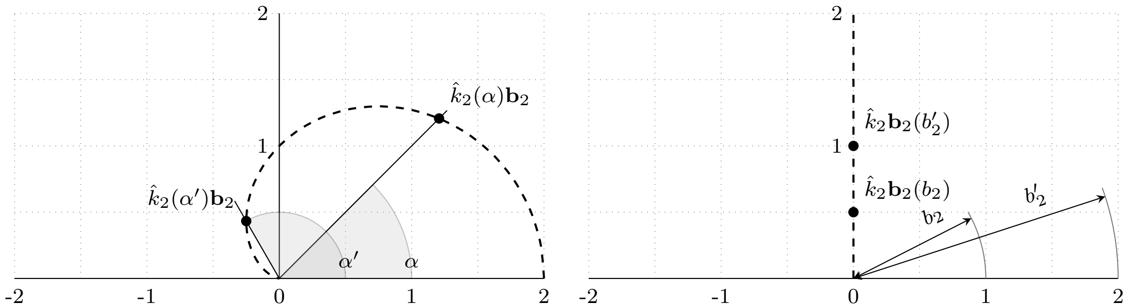
[14] The size of the set of initial incumbent's policy commitments for which this is true increases in b_2 and decreases in α. In addition, for any $\mathbf{q}^0 \neq \mathbf{b}_v$, it is possible to find (α, δ) sufficiently close to $(\pi, 1)$ such that $\mathbf{q}^0$ is sufficiently far from $\mathbf{b}_v$ to generate interesting policy dynamics.

[15] To keep the figures simple, we do not show the long-run policy when party 1 is in power, $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$. This policy lies on the horizontal axis, at the same distance from the origin as $\hat{k}_2\mathbf{b}_2$.

[16] The only other policy that can be observed in equilibrium is $\mathbf{b}_1$, the policy platform party 1 chooses when at the opposition in the first period. While α has no impact on this transient policy, the larger is b_1, the further away is this transient policy from the origin and, in turn, the worse off is the representative voter.

Figure 5: $\hat{k}_2\mathbf{b}_2$ as a function of α and b_2
$\delta \to 1$ and $b_1 > b_2$

(a) Effect of α for $b_2 = 2$ (b) Effect of b_2 for $\alpha = \frac{1}{2}\pi$



the extremism of party 2, b_2, on the other hand, has the opposite effect (Figure 5b). The strength of the moderating force does not change with b_2 and, thus, $\hat{k}_2$ is unchanged. Since the party's bliss point has moved further away from the origin and the weight applied to it has not changed, $\hat{k}_2\mathbf{b}_2$ moves away from the origin too. Since the representative voter's utility depends on the distance of the equilibrium policies from her bliss point, this has a negative effect on her welfare. Why does $\hat{k}_2$ depend on α but not on b_2 (or b_1)? Increasing extremism increases both the marginal benefit of moderation (that is, the gain from constraining the future incumbent to more moderate policies) and the marginal cost of moderation (that is, the loss from implementing policies further away from the bliss point), *leaving their ratio constant.* On the other hand, increasing antagonism has no influence on the marginal costs of moderation but it increases its marginal benefits.[17,18]

The next result presents the marginal impact of antagonism and extremism on the variance of the long-run policy outcomes. This is an interesting prediction as it allows us to link preferences' disagreement to policy volatility. Denote with $d_p =$

[17] To see that the ratio of the marginals does not change with b_2, consider party 2 moderating to $k\mathbf{b}_2$ and hence constraining party 1 to $k_1(k\mathbf{b}_2)\mathbf{b}_1$. The utility of party 2 from these two policies is, using standard trigonometry, $-b_2^2(1-k)^2$ and $-b_2^2((\cos\alpha - k)^2 + \sin^2\alpha)$, respectively.

[18] Figure 5 shows that the equilibrium policies do not converge to the representative voter's bliss point even when the parties become arbitrarily patient. This contrasts with convergence results in the literature on dynamic elections with adverse selection and/or moral hazard (see Duggan and Martinelli, 2015, for a thorough discussion) or in dynamic legislative bargaining models with endogenous status-quo (for example Baron, 1996). Convergence in these models is driven by candidates or proposers that share the representative voter's preferences. Policies do not converge in our model because the marginal costs and marginal benefits of moderation are proportional to $1-k$ and $k - \cos\alpha$, respectively, where the latter term only accrues in the future and, thus, is discounted by δ. Convergence corresponds to the case $k = 0$. When $k = 0$, the marginal costs are larger than the marginal benefits unless $\alpha = \pi$ and $\delta = 1$. Informally, policies do not converge because party i, by moving its own policy ϵ closer to $\mathbf{b}_v$ and, hence, ϵ further away from $\mathbf{b}_i$, moves its opponent's policy closer to $\mathbf{b}_i$ by less than ϵ.

$d(\hat{k}_2\mathbf{b}_2, k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1)$ the distance between the two long-run policies in the unique SSMPE from Proposition 1. This measure determines the variance of the long-run equilibrium policies characterized above.

Proposition 3. *Assume that $b_1 > b_2$. d_p is:*

1. *non-decreasing in b_2; increasing in b_2 if $\alpha > 0$; constant in b_1;*

2. *increasing in α when $\alpha \in [0, \alpha']$ and decreasing in α when $\alpha \in [\alpha', \pi]$, where $\alpha' \in [0, \pi]$ and $\alpha' < \pi \Leftrightarrow \delta > \frac{1}{5}$.*

Proof. See Appendix A1.

Figure 6: d_p as a function of α and b_2
$\delta \to 1$ and $b_1 > b_2$

(a) Effect of α for $b_2 = 2$

(b) Effect of b_2 for $\alpha = \frac{1}{2}\pi$

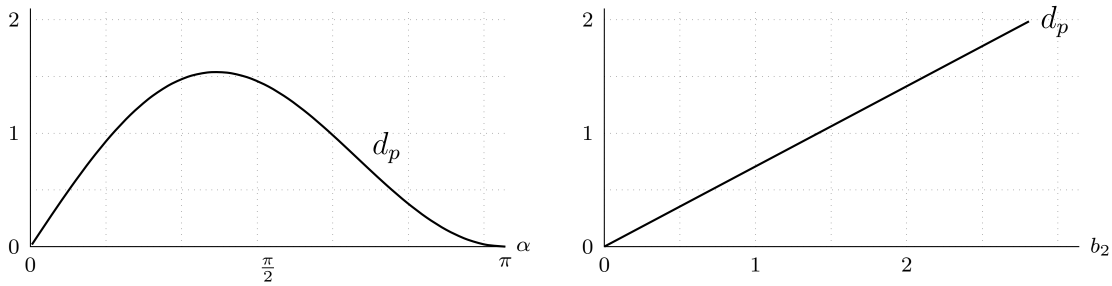


Figure 6 shows how the long-run policies variance change when we increase α (Figure 6a) and b_2 (Figure 6b), in the limit as $\delta \to 1$. Increasing the extremism of party 2 has a straightforward effect on policy volatility, as it pushes the implemented policies away from the representative voter's ideal point in different directions. Increasing the extremism of party 1, on other hand, has no effect because the long-run equilibrium policies depend only on the degree of moderation of party 2.[19] Finally, increasing antagonism has two effects: it distances the long-run policies away from each other but it also moves them closer to the ideal point of the representative voter. The former effects dominates for low α, while the latter dominates for high α.

[19] Proposition 3 assumes $b_1 > b_2$. If $b_1 = b_2$, part two holds with no change. In this case, any increase in b_2 means that party 1 moderates in equilibrium. If we define d_p using the equilibrium extent of moderation by party 1, part one holds except for switching b_1 and b_2.

7 Discussion and Conclusions

While many commentators and scholars diagnose a sharp and increasing ideological divide between political elites in the U.S. and other mature democracies, both the popular press and the existing literature are somewhat unclear about what exactly constitutes polarization and how one can measure this concept.

In this paper, we study an environment where two ideological and forward-looking parties compete for office in a sequence of elections. We assume that incumbents who are reelected find it too costly to implement policies that differ from those of their first term. On the other hand, challengers are free to offer to the representative voter any policy platform. We use two different measures to describe the political environment and the degree of conflict among agents' preferences. The first measure, which we label *extremism*, is the ideological distance of each party from the representative voter. The second measure, which we call *antagonism*, is the ideological distance that separates the two parties from each other and summarizes the degree of political competition between policymakers. These two measures coincide in a one dimensional policy space, where the ideological distance between the two parties can increase only as they move further away from a moderate representative voter. However, they do not coincide in a two-dimensional setting: here, the two parties can be very close—when they share views on both dimensions—or very different—when they are perfectly opposed in one dimension—without altering their overall distance from the representative voter.

We show that a stationary Markov perfect equilibrium of this game exists and we fully characterize it for any discount factor, initial incumbent's policy, and degree of extremism and antagonism. In this equilibrium, increasing the degree of extremism reduces the welfare of the representative voter; on the other hand, increasing antagonism increases the influence exerted by the electorate on long run policy outcomes.[20]

To conclude, we discuss the methodological and empirical implications of our model. Our analysis suggests that, in order to understand whether the electorate is

[20] While there might exist other equilibria, with multiple moderating steps (akin to those in Forand, 2014) or in mixed strategies (akin to those in Kalandrakis, 2016), our welfare implications of antagonism and extremism should carry beyond the equilibria in simple strategies; antagonism drives the incentive to moderate and its increase will strengthen the incentive in general while extremism disassociates parties and voters. Moreover, the representative voter's welfare derived is likely to be a lower bound across (potential) other equilibria; it is based on a single moderating step and support of any mixed strategy has to be over policies the voter prefers to the current incumbent policy commitment.

better or worse off with increasingly polarized parties, social scientists need to be careful about how they measure polarization. Observing that parties' ideal points are moving further away from each other does not necessarily imply that voters will suffer. In fact, this crucially depends on the distance between the parties' and the voters' ideal points. It is, thus, important to gauge both measures.

The existing empirical literature on ideological polarization has mostly measured the ideal points of parties and has ignored the electorate. Poole and Rosenthal (1997) used roll call voting choices and item response models to estimate the relative ideological leanings of U.S. Congressmen, yielding what are called NOMINATE scores. Similar techniques were then used to estimate multidimensional ideal points of European parties, U.S. state legislators, and other elected officials. In recent years, Bayesian approaches have been applied to the problem of recovering preferences from observed choices, electoral platforms, or election results.[21]

A key limitation of these estimations comes from the fact that ideal points are only defined relatively and two separate sets of ideal point estimates are, in general, not comparable. Our theoretical approach suggests that, to evaluate the policy and welfare consequences of polarization, we need ideal point estimates for both elected officials and voters, and, importantly, that we need these ideal points to be comparable or, in other words, to reside in a common policy space. The measures mentioned above, for example the NOMINATE scores, do not do this and, focusing on politicians' ideal points, they are likely to capture what we label as antagonism, ignoring extremism.

As a consequence, the increasing distance between the main American parties on these measures—for example, the divergence in NOMINATE scores between the median Republican and the median Democrat in Congress—does not necessarily imply more extreme policies nor that the electorate is worse off. If parties are moving further away from each other *and* from the electorate, then increased elite polarization coincides with increased extremism, using the language of our model, and this is likely bad news for the moderate voters: as polarization increases, the observed policies will be more extreme, even when parties are forward looking and policy motivated. On the other hand, if the increased ideological conflict between parties is associated with an unchanged distance from the median voter's preferences—which can happen when the ideological conflict among actors is on multiple dimensions—this might be good news for moderate voters: the heightened competition between parties will not favor the civility of the political discourse and might lead to pol-

[21] See Laver (2014) for a detailed survey of this literature.

icy gridlock, but forward looking and policy motivated parties will propose more moderate policies. It is not possible to distinguish between these two predictions without measuring jointly the ideological positions of the parties and the electorate.

In recent years, political scientists have started to bridge this gap and have attempted to measure the ideological position of elected leaders and members of the public on a common policy space (see, for example, Jessee, 2009, 2010; Bafumi and Herron, 2010; Bonica, 2014; Lo, Proksch, and Gschwend, 2014; Barbera, 2015). While these are important advances, the existing works suffer from some important shortcomings that hinder our ability to evaluate the normative consequences of polarization through the lenses of our theoretical results: most of these studies rely on survey questions to gauge the ideal points of voters and assume these responses are directly comparable with roll call votes on the same issues; more importantly, they assume that policies and preferences are unidimensional—making it impossible to disentangle antagonism and extremism—and focus on a short time span (a single election or, at most, two U.S. Congresses).

Our theoretical analysis supports this recent direction and highlights the importance for future empirical research in legislative studies and political behavior of exploring multiple policy dimensions and time trends. Over time, the research program being advanced by these scholars, together with the predictions of dynamic models of elections, can be used to provide a nuanced perspective on the impact of elite polarization on implemented policies and electorate's welfare.

References

Adams, J., M. Clark, L. Ezrow, and G. Glasgow (2006). Are niche parties fundamentally different from mainstream parties? The causes and the electoral consequences of Western European parties' policy shifts, 1976-1998. *American Journal of Political Science 50*(3), 513–529.

Albright, J. J. (2010). The multidimensional nature of party competition. *Party Politics 16*(6), 699–719.

Aldrich, J. H., D. W. Rohde, and M. W. Tofias (2007). One D is not enough: Measuring conditional party government, 1887-2002. In D. W. Brady and M. D. McCubbins (Eds.), *Party, Process, and Political Change in Congress, Volume 2: Further New Perspectives on the History of Congress*, pp. 102–112. Stanford, CA: Stanford University Press.

Alesina, A. and G. Tabellini (1990). A positive theory of fiscal deficits and government debt. *Review of Economic Studies 57*(3), 403–414.

Azzimonti, M. (2011). Barriers to investment in polarized societies. *American Economic Review 101*(5), 2182–2204.

Bafumi, J. and M. C. Herron (2010). Leapfrog representation and extremism: A study of American voters and their members in Congress. *American Political Science Review 104*(3), 519–542.

Bai, J. H. and R. Lagunoff (2011). On the Faustian dynamics of policy and political power. *Review of Economic Studies 78*(1), 17–48.

Banks, J. S. and J. Duggan (2006). A social choice lemma on voting over lotteries with applications to a class of dynamic games. *Social Choice and Welfare 26*(2), 285–304.

Banks, J. S. and J. Duggan (2009). A dynamic model of democratic elections in multidimensional policy spaces. *Quarterly Journal of Political Science 3*(3), 269–299.

Barbera, P. (2015). Birds of the same feather tweet together: Bayesian ideal point estimation using twitter data. *Political Analysis 23*(1), 76–91.

Baron, D. P. (1996). A dynamic theory of collective goods programs. *American Political Science Review 90*(2), 316–330.

Baron, D. P., D. Diermeier, and P. Fong (2012). A dynamic theory of parliamentary democracy. *Economic Theory 49*(3), 703–738.

Battaglini, M. (2014). A dynamic theory of electoral competition. *Theoretical Economics 9*(2), 515–554.

Battaglini, M., S. Nunnari, and T. R. Palfrey (2012). Legislative bargaining and the dynamics of public investment. *American Political Science Review 106*(2), 407–429.

Bernhardt, D., L. Campuzano, F. Squintani, and O. Camara (2009). On the benefits of party competition. *Games and Economic Behavior 66*(2), 685–707.

Bernhardt, D., S. Dubey, and E. Hughson (2004). Term limits and pork barrel politics. *Journal of Public Economics 88*(12), 2383–2422.

Bernhardt, D., J. Duggan, and F. Squintani (2009). The case for responsible parties. *American Political Science Review 103*(4), 570–587.

Bonica, A. (2014). Mapping the ideological marketplace. *American Journal of Political Science 58*(2), 367–386.

Bowen, R. T., Y. Chen, and H. Eraslan (2014). Mandatory versus discretionary spending: The status quo effect. *American Economic Review 104*(10), 2941–2974.

Brown, D. S., M. Touchton, and A. Whitford (2011). Political polarization as a constraint on corruption: A cross-national comparison. *World Development 39*(9), 1516–1529.

Budge, I., H. D. Klingemann, A. Volkens, J. Bara, and E. Tanenbaum (2001). *Mapping Policy Preferences: Estimates for Parties, Electors, and Governments 1945-1998*. Oxford: Oxford University Press.

Cox, G. W. and M. D. McCubbins (1993). *Legislative Leviathan: Party Government in the House*. Los Angeles, CA: University of California Press.

Cox, G. W. and M. D. McCubbins (2005). *Setting the Agenda: Responsible Party Government in the US House of Representatives*. New York, NY: Cambridge University Press.

DeBacker, J. M. (2015). Flip-flopping: Ideological adjustment costs in the United States Senate. *Economic Inquiry 53*(1), 108–128.

Diermeier, D. and P. Fong (2011). Legislative bargaining with reconsideration. *Quarterly Journal of Economics 126*(2), 947–985.

Duggan, J. (2000). Repeated elections with asymmetric information. *Economics & Politics 12*(2), 109–135.

Duggan, J. (2012). A survey of equilibrium analysis in spatial models of elections. *Unpublished Manuscript*.

Duggan, J. and J. G. Forand (2013). Markovian elections. *Unpublished Manuscript*.

Duggan, J. and A. Kalandrakis (2012). Dynamic legislative policy making. *Journal of Economic Theory 147*(5), 1653–1688.

Duggan, J. and C. Martinelli (2015). The political economy of dynamic elections: Accountability, commitment, and responsiveness. *forthcoming in Journal of Economic Literature*.

Dziuda, W. and A. Loeper (2016). Dynamic collective choice with endogenous status quo. *Journal of Political Economy 124*(4), 1148–1186.

Fiorina, M. P. (1981). *Retrospective Voting in American National Elections.* New Haven, CT: Yale University Press.

Fiorina, M. P. and S. J. Abrams (2008). Political polarization in the American public. *Annual Review of Political Science 11*, 563–588.

Forand, J. G. (2014). Two-party competition with persistent policies. *Journal of Economic Theory 152*, 64–91.

Frye, T. (2002). The perils of polarization. *World Politics 54*(3), 308–337.

Hetherington, M. J. (2001). Resurgent mass partisanship: The role of elite polarization. *American Political Science Review 95*(3), 619–632.

Janda, K., R. Harmel, C. Edens, and P. Goff (1995). Changes in party identity: Evidence from party manifestos. *Party Politics 1*(2), 171–196.

Jessee, S. A. (2009). Spatial voting in the 2004 presidential election. *American Political Science Review 103*(1), 59–81.

Jessee, S. A. (2010). Partisan bias, political information and spatial voting in the 2008 presidential election. *Journal of Politics 72*(2), 327–340.

Kalandrakis, A. (2004). A three-player dynamic majoritarian bargaining game. *Journal of Economic Theory 116*(2), 294–322.

Kalandrakis, A. (2009). A reputational theory of two-party competition. *Quarterly Journal of Political Science 4*(4), 343–378.

Kalandrakis, A. (2010). Minimum winning coalitions and endogenous status quo. *International Journal of Game Theory 39*(4), 617–643.

Kalandrakis, A. (2016). Pareto efficiency in the dynamic one-dimensional bargaining model. *Journal of Theoretical Politics 28*(4), 525–536.

Klašnja, M. (2016). Increasing rents and incumbency disadvantage. *Journal of Theoretical Politics 28*(2), 225–265.

Klašnja, M. and R. Titiunik (2017). The incumbency curse: Weak parties, term limits, and unfulfilled accountability. *American Political Science Review 111*(1), 129–148.

Klingemann, H. D., A. Volkens, J. Bara, I. Budge, and M. D. McDonald (2006). *Mapping Policy Preferences II: Estimates for Parties, Electors, and Governments in Eastern Europe, European Union, and OECD 1990-2003*. Oxford: Oxford University Press.

Kollman, K. W., J. H. Miller, and S. E. Page (1998). Political parties and electoral landscapes. *British Journal of Political Science 28*(1), 139–158.

Kramer, G. H. (1977). A dynamical model of political equilibrium. *Journal of Economic Theory 16*(2), 310–334.

Krasa, S. and M. Polborn (2014). Social ideology and taxes in a differentiated candidates framework. *American Economic Review 104*(1), 308–322.

Krusell, P. and J.-V. Rios-Rull (1999). On the size of US government: Political economy in the neoclassical growth model. *American Economic Review 89*(5), 1156–1181.

Laver, M. (2014). Measuring policy positions in political space. *Annual Review of Political Science 17*(1), 207–223.

Layman, G. C. and T. M. Carsey (2002). Party polarization and party structuring of policy attitudes: A comparison of three NES panel studies. *Political Behavior 24*(3), 199–236.

Linz, J. J. and A. C. Stepan (1978). *The Breakdown of Democratic Regimes, Latin America*. Baltimore, MD: Johns Hopkins University Press.

Lo, J., S.-O. Proksch, and T. Gschwend (2014). A common left-right scale for voters and parties in Europe. *Political Analysis 22*(2), 205–223.

Maskin, E. and J. Tirole (2001). Markov perfect equilibrium: I Observable actions. *Journal of Economic Theory 100*(2), 191–219.

McCarty, N. M., K. T. Poole, and H. Rosenthal (2006). *Polarized America: The Dance of Ideology and Unequal Riches*. Cambridge, MA: MIT Press.

Miller, G. and N. Schofield (2003). Activists and partisan realignment in the United States. *American Political Science Review 97*(2), 245–260.

Miller, G. and N. Schofield (2008). The transformation of the Republican and Democratic party coalitions in the U.S. *Perspectives on Politics 6*(3), 433–450.

Persson, T. and L. Svensson (1989). Why a stubborn conservative would run a deficit: Policy with time-inconsistent preferences. *Quarterly Journal of Economics 104*(2), 325–345.

Plott, C. R. (1967). A notion of equilibrium and its possibility under majority rule. *American Economic Review 57*(4), 787–806.

Poole, K. T. and H. Rosenthal (1997). *Congress: A Political-Economic History of Roll Call Voting*. New York, NY: Oxford University Press.

Powell, Jr., G. B. and G. D. Whitten (1993). A cross-national analysis of economic voting: Taking account of the political context. *American Journal of Political Science 37*(2), 391–414.

Prato, C. (2016). Electoral competition and partisan policy feedbacks. *forthcoming in Journal of Politics*.

Rose, R. and T. T. Mackie (1983). Incumbency in government: Asset or liability? In H. Daalder and P. Mair (Eds.), *Western European Party Systems: Continuity & Change*, pp. 115–137. London: Sage Publications.

Schofield, N., G. Miller, and A. Martin (2003). Critical elections and political realignments in the USA: 1860-2000. *Political Studies 51*(2), 217–240.

Shor, B. and N. McCarty (2011). The ideological mapping of American legislatures. *American Political Science Review 105*(3), 530–551.

Somer-Topcu, Z. (2009). Timely decisions: The effects of past national elections on party policy change. *Journal of Politics 71*(1), 238–248.

Stoll, H. (2010). Elite-level conflict salience and dimensionality in Western Europe: Concepts and empirical findings. *West European Politics 33*(3), 445–473.

Strom, K. (1985). Party goals and government performance in parliamentary democracies. *American Political Science Review 79*(3), 738–754.

Tavits, M. (2007). Principle vs. pragmatism: Policy shifts and political competition. *American Journal of Political Science 51*(1), 151–165.

Tomz, M. and R. P. Van Houweling (2008). Candidate positioning and voter choice. *American Political Science Review 102*(3), 303–318.

Tomz, M. and R. P. Van Houweling (2012). Candidate repositioning. *Unpublished Manuscript*.

Uppal, Y. (2009). The disadvantaged incumbents: Estimating incumbency effects in Indian state legislatures. *Public Choice 138*(1-2), 9–27.

Van Weelden, R. (2013). Candidates, credibility, and re-election incentives. *Review of Economic Studies 80*(4), 1622–1651.

Van Weelden, R. (2015). The welfare implications of electoral polarization. *Social Choice and Welfare 45*(4), 653–686.

Veiga, F. J. and L. G. Veiga (2004). Popularity functions, partisan effects, and support in parliament. *Economics & Politics 16*(1), 101–115.

Wittman, D. (1977). Candidates with policy preferences: A dynamic model. *Journal of Economic Theory 14*(1), 180–189.

A1 Proofs

Proof of Proposition 1

Let $\hat{k}_1$ and $\hat{k}_2$ be the parameters associated with simple strategies and let $\hat{k}_1^*$ and $\hat{k}_2^*$ be the parameters associated with the simple strategies of a SSMPE (as described in Definition 3). When talking about both parameters jointly, we use $\hat{k} = (\hat{k}_1, \hat{k}_2)$ and $\hat{k}^* = (\hat{k}_1^*, \hat{k}_2^*)$. Because a simple strategy of $i \in \{1, 2\}$ from Definition 3 is fully determined by $\hat{k}_i$, with a slight abuse of notation, we call $\hat{k}_i$ the strategy of i and $\hat{k} = (\hat{k}_1, \hat{k}_2)$ the profile of strategies.

We start by noting that any SSMPE strategy of v satisfies, for an incumbent $j \in \{1, 2\}$, its policy commitment $\mathbf{q} \in X$ and a platform of the opposition $\mathbf{p} \in X$, $\sigma_v^*(j, \mathbf{q}, \mathbf{p}) = Yes$ if and only if $d(\mathbf{p}) \leq d(\mathbf{q})$. From Definition 2, $\sigma_v^*(j, \mathbf{q}, \mathbf{p}) = Yes$ if and only if $u_v(\mathbf{p}) \geq u_v(\mathbf{q})$, which, from $u_v(\mathbf{x}) = -d^2(\mathbf{x})$ for any $\mathbf{x} \in X$, is equivalent to $d(\mathbf{p}) \leq d(\mathbf{q})$. Because of the simplicity of the representative voter's behavior in any SSMPE, we suppress σ_v from the notation below.

We now claim that, for any SSMPE strategy of v and any simple strategy profile of the two parties $\hat{k}$, the opposition party contests elections and wins. The former follows from $\mathbf{p}_i(\mathbf{q}) \neq Out$ for any $i \in \{1, 2\}$, $\hat{k}_i \in \mathbb{R}$ and $\mathbf{q} \in X$. To prove the latter claim, we need to show that $d(\mathbf{p}_i(\mathbf{q})) \leq d(\mathbf{q})$ for any $i \in \{1,2\}$, $\hat{k}_i \in \mathbb{R}$ and $\mathbf{q} \in X$. From Definition 3, if $k_i(\mathbf{q}) \leq \hat{k}_i$, then $d(\mathbf{p}_i(\mathbf{q})) = d(k_i(\mathbf{q})\mathbf{b}_i) = k_i(\mathbf{q})b_i = \frac{d(\mathbf{q})}{b_i}b_i$ and if $k_i(\mathbf{q}) \geq \hat{k}_i$, then $d(\mathbf{p}_i(\mathbf{q})) = d(\hat{k}_i\mathbf{b}_i) = \hat{k}_i b_i \leq k_i(\mathbf{q})b_i = d(\mathbf{q})$. Because the opposition party contests elections and wins for any SSMPE strategy of v and any simple strategy profile $\hat{k}$, elections are always contested and incumbents always defeated in any SSMPE. This proves part 3 of the proposition.

We now prove parts 1 (characterization) and 2 (uniqueness or duplicity) of the proposition. By the one-stage-deviation principle, a profile of strategies $\hat{k}^*$ constitutes a SSMPE if, for any $i \in \{1,2\}$ and $\mathbf{q} \in X$, a deviation by the opposition party i to contest elections with $\mathbf{p} \neq \mathbf{p}_i(\mathbf{q})$ or to stay out is not profitable. We momentarily assume and later verify that both parties, when at the opposition, want to contest elections with platforms that guarantee their victory.

For any $\hat{k}$, $i \in \{1,2\}$ and $\mathbf{q} \in X$, the dynamic utility i receives from running with $\mathbf{p}$ such that $d(\mathbf{p}) \leq d(\mathbf{q})$ is $u_i(\mathbf{p}) + r + \delta V_i^i(\mathbf{p}|\hat{k})$. An opposition party i thus runs with $\mathbf{p}^* \in \arg\max_{d(\mathbf{x}) \leq d(\mathbf{q})} u_i(\mathbf{x}) + r + \delta V_i^i(\mathbf{x}|\hat{k})$. For any $\hat{k}$ and any two policies $\mathbf{p} \in X$ and $\mathbf{p}' \in X$ with $d(\mathbf{p}) = d(\mathbf{p}')$, we have $V_i^i(\mathbf{p}|\hat{k}) = V_i^i(\mathbf{p}'|\hat{k})$. This means that the optimal $\mathbf{p}^*$ has to lie on the ray starting at $\mathbf{b}_v = (0,0)$ and passing through $\mathbf{b}_i$. It hence can be written as $\mathbf{p}^* = k\mathbf{b}_i$ for some $k \geq 0$. Denote by $U_i(k|\hat{k}) = u_i(k\mathbf{b}_i) + r + \delta V_i^i(k\mathbf{b}_i|\hat{k})$ the dynamic utility of party $i \in \{1,2\}$ from running with policy $k\mathbf{b}_i$ when the parties use simple strategies characterized by $\hat{k}$. The key properties of $U_i(k|\hat{k})$ are summarized in the lemma below. To facilitate its proof, we first state several identities for the policy utility of the two parties.

The policy utility $i \in \{1,2\}$ derives from $k\mathbf{b}_j$ where $k \geq 0$ and $j \in \{1,2\}$ is $u_i(k\mathbf{b}_j) = -d^2(k\mathbf{b}_j, \mathbf{b}_i)$. This is a continuous and differentiable function of k. Using $-i = \{1,2\} \setminus \{i\}$, we have

$$\begin{aligned}
\frac{\partial(-d^2(k\mathbf{b}_{-i}, \mathbf{b}_i))}{\partial k} &= -2(kd^2(\mathbf{b}_{-i}) - \mathbf{b}_{-i} \cdot \mathbf{b}_i) = -2b_i b_{-i}\left(k\frac{b_{-i}}{b_i} - \cos\alpha\right) \\
\frac{\partial(-d^2(k\mathbf{b}_i, \mathbf{b}_i))}{\partial k} &= -2(kd^2(\mathbf{b}_i) - \mathbf{b}_i \cdot \mathbf{b}_i) = -2b_i^2(k-1) \\
\frac{\partial^2(-d^2(k\mathbf{b}_j, \mathbf{b}_i))}{\partial^2 k} &= -2d^2(\mathbf{b}_j).
\end{aligned} \tag{A1}$$

This proves that $u_i(k\mathbf{b}_j)$ is a concave function of k.

Lemma A1. *For any $i \in \{1,2\}$, fix $\hat{k}_i \in \mathbb{R}$ and $\hat{k}_{-i} \in \mathbb{R}$. Let $\mathbf{q} \in X$ be the incumbent's policy commitment. $U_i(k|\hat{k})$, as a function of $k \in [0, k_i(\mathbf{q})]$, is continuous, differentiable except when $k = \hat{k}_i$ or $k = \hat{k}_{-i}\frac{b_{-i}}{b_i}$, strictly concave on each interval* *on which it is differentiable and* $\frac{\partial U_i(k|\hat{k})}{\partial k} > 0$ *for* $k < c$, $\frac{\partial U_i(k|\hat{k})}{\partial k} = 0$ *for* $k = c$ *and* $\frac{\partial U_i(k|\hat{k})}{\partial k} < 0$ *for* $k > c$ *where*

$$\begin{array}{lll} c = \frac{1+\delta\cos\alpha}{1+\delta} & \textit{if} & k < \min\{\hat{k}_i, \hat{k}_{-i}\frac{b_{-i}}{b_i}\} \\ c = \frac{1+\delta\cos\alpha}{1+\delta} & \textit{if} & \hat{k}_i < k < \hat{k}_{-i}\frac{b_{-i}}{b_i} \\ c = 1 & \textit{if} & \hat{k}_{-i}\frac{b_{-i}}{b_i} < k < \hat{k}_i \\ c = 1 & \textit{if} & \max\{\hat{k}_i, \hat{k}_{-i}\frac{b_{-i}}{b_i}\} < k. \end{array} \tag{A2}$$

Proof. Throughout the proof fix $i \in \{1,2\}$, $-i = \{1,2\} \setminus \{i\}$, $\hat{k}_i \in \mathbb{R}$, $\hat{k}_{-i} \in \mathbb{R}$ and incumbent's policy commitment $\mathbf{q} \in X$. The continuity is easy to see as the simple strategies characterized by $\hat{k}$ give rise to a value function, $V_i^i(\mathbf{x}|\hat{k})$, that is continuous in $\mathbf{x} \in X$ for any $d(\mathbf{x}) \leq d(\mathbf{q})$. For the remaining properties, we have to derive $V_i^i(\mathbf{x}|\hat{k})$ explicitly. Because the opposition party always contests and wins elections for any $\hat{k}$, we have, for any $\mathbf{x} \in X$ with $d(\mathbf{x}) \leq d(\mathbf{q})$,

$$\begin{aligned} V_i^{-i}(\mathbf{x}|\hat{k}) &= u_i(\mathbf{p}_i(\mathbf{x})) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{x})|\hat{k}) \\ V_i^i(\mathbf{x}|\hat{k}) &= u_i(\mathbf{p}_{-i}(\mathbf{x})) + \delta V_i^{-i}(\mathbf{p}_{-i}(\mathbf{x})|\hat{k}). \end{aligned} \tag{A3}$$

Combining the two equations

$$V_i^i(\mathbf{x}|\hat{k}) = u_i(\mathbf{p}_{-i}(\mathbf{x})) + \delta\left[u_i(\mathbf{p}_i(\mathbf{p}_{-i}(\mathbf{x}))) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{p}_{-i}(\mathbf{x}))|\hat{k})\right] \tag{A4}$$

where

$$\begin{array}{lll} \mathbf{p}_{-i}(\mathbf{x}) = k_i(\mathbf{x})\frac{b_i}{b_{-i}}\mathbf{b}_{-i} & \mathbf{p}_i(\mathbf{p}_{-i}(\mathbf{x})) = k_i(\mathbf{x})\mathbf{b}_i & \text{if } k_i(x) \leq \min\{\hat{k}_i, \hat{k}_{-i}\frac{b_{-i}}{b_i}\} \\ \mathbf{p}_{-i}(\mathbf{x}) = k_i(\mathbf{x})\frac{b_i}{b_{-i}}\mathbf{b}_{-i} & \mathbf{p}_i(\mathbf{p}_{-i}(\mathbf{x})) = \hat{k}_i\mathbf{b}_i & \text{if } \hat{k}_i < k_i(x) < \hat{k}_{-i}\frac{b_{-i}}{b_i} \\ \mathbf{p}_{-i}(\mathbf{x}) = \hat{k}_{-i}\mathbf{b}_{-i} & \mathbf{p}_i(\mathbf{p}_{-i}(\mathbf{x})) = \hat{k}_{-i}\frac{b_{-i}}{b_i}\mathbf{b}_i & \text{if } \hat{k}_{-i}\frac{b_{-i}}{b_i} < k_i(x) < \hat{k}_i \\ \mathbf{p}_{-i}(\mathbf{x}) = \hat{k}_{-i}\mathbf{b}_{-i} & \mathbf{p}_i(\mathbf{p}_{-i}(\mathbf{x})) = \min\{\hat{k}_{-i}\frac{b_{-i}}{b_i}, \hat{k}_i\}\mathbf{b}_i & \text{if } \max\{\hat{k}_i, \hat{k}_{-i}\frac{b_{-i}}{b_i}\} \leq k_i(x) \end{array} \tag{A5}$$

is easy to confirm using properties of the simple strategies along with $k_i(\mathbf{x})b_i = k_{-i}(\mathbf{x})b_{-i}$ for any $\mathbf{x} \in X$.

We substitute $\mathbf{x} = k\mathbf{b}_i$ into (A4), using (A5), $k_i(k\mathbf{b}_i) = k$, $u_i(\mathbf{x}) = -d^2(\mathbf{x}, \mathbf{b}_i)$ and $V_i^i(\mathbf{x}|\hat{k}) = V_i^i(k_i(\mathbf{x})\mathbf{b}_i|\hat{k})$, which follows from $d(k_i(\mathbf{x})\mathbf{b}_i) = \frac{d(\mathbf{x})}{b_i}d(\mathbf{b}_i) = d(\mathbf{x})$. After some straightforward algebra, summarizing with χ_t all the terms constant in

k, we have:

$$\begin{aligned}
U_i(k|\hat{k}) &= \frac{-d^2(k\mathbf{b}_i,\mathbf{b}_i)-d^2(k\frac{b_i}{b_{-i}}\mathbf{b}_{-i},\mathbf{b}_i)\delta+r}{1-\delta^2} && \text{if } k\le \min\{\hat{k}_i,\hat{k}_{-i}\tfrac{b_{-i}}{b_i}\}\\
U_i(k|\hat{k}) &= -d^2(k\mathbf{b}_i,\mathbf{b}_i)-d^2(k\tfrac{b_i}{b_{-i}}\mathbf{b}_{-i},\mathbf{b}_i)\delta+\chi_1 && \text{if } \hat{k}_i<k<\hat{k}_{-i}\tfrac{b_{-i}}{b_i}\\
U_i(k|\hat{k}) &= -d^2(k\mathbf{b}_i,\mathbf{b}_i)+\chi_2 && \text{if } \hat{k}_{-i}\tfrac{b_{-i}}{b_i}<k<\hat{k}_i\\
U_i(k|\hat{k}) &= -d^2(k\mathbf{b}_i,\mathbf{b}_i)+\chi_3 && \text{if } \max\{\hat{k}_i,\hat{k}_{-i}\tfrac{b_{-i}}{b_i}\}\le k
\end{aligned} \tag{A6}$$

for any $k\in[0,k_i(\mathbf{q})]$. Direct verification then shows all the remaining properties of $U_i(k|\hat{k})$.[22] $\square$

Using $U_i(k|\hat{k})$ we can rewrite the optimization problem of opposition party $i\in\{1,2\}$ regarding which policy to contest elections with, for any incumbent's policy commitment $\mathbf{q}\in X$, as $\max_{0\le k\le k_i(\mathbf{q})} U_i(k|\hat{k})$. To find a SSMPE, we need to find a $\hat{k}^*$ such that the solution to this optimization problem under $\hat{k}^*$, for any $\mathbf{q}\in X$, can be described by $\hat{k}^*$.

We first claim that in any SSMPE, $\hat{k}^*_i\in\{\frac{1+\delta\cos\alpha}{1+\delta},1\}$ for $i\in\{1,2\}$. Notice $\frac{1+\delta\cos\alpha}{1+\delta}\le 1$ for any $\delta\in[0,1)$ and $\alpha\in[0,\pi]$. To show the claim, suppose, towards a first contradiction, that $\hat{k}^*_i<\frac{1+\delta\cos\alpha}{1+\delta}$. Suppose $\hat{k}^*_{-i}\frac{b_{-i}}{b_i}\le\hat{k}^*_i$. Then from Lemma A1, there exists $\bar{\epsilon}>0$ such that for all $\epsilon\in(0,\bar{\epsilon})$, $U_i'(\hat{k}^*_i+\epsilon|\hat{k}^*)>0$, which implies that $U_i(\hat{k}^*_i|\hat{k}^*)<U_i(\hat{k}^*_i+\epsilon|\hat{k}^*)$, $\mathbf{p}_i((\hat{k}^*_i+\epsilon)\mathbf{b}_i)=\hat{k}^*_i\mathbf{b}_i$ and $d((\hat{k}^*_i+\epsilon)\mathbf{b}_i)<d((\hat{k}^*_i+\bar{\epsilon})\mathbf{b}_i)$. In words, for incumbent's policy commitment $(\hat{k}^*_i+\bar{\epsilon})\mathbf{b}_i$, i contests elections with $\hat{k}^*_i\mathbf{b}_i$, despite the fact that running with $(\hat{k}^*_i+\epsilon)\mathbf{b}_i$ would ensure its victory and higher dynamic utility, a contradiction. An identical argument leads to a contradiction when $\hat{k}^*_{-i}\frac{b_{-i}}{b_i}>\hat{k}^*_i$. Now suppose, towards a second contradiction, that $\hat{k}^*_i\in(\frac{1+\delta\cos\alpha}{1+\delta},1)$. If $\hat{k}^*_{-i}\frac{b_{-i}}{b_i}\le\hat{k}^*_i$, an argument identical to the one above leads to a contradiction. Suppose $\hat{k}^*_{-i}\frac{b_{-i}}{b_i}>\hat{k}^*_i$. Then from Lemma A1, there exists $\bar{\epsilon}>0$ such that for all $\epsilon\in(0,\bar{\epsilon})$, $U_i'(\frac{1+\delta\cos\alpha}{1+\delta}+\epsilon|\hat{k}^*)<0$, which implies that $U_i(\frac{1+\delta\cos\alpha}{1+\delta}|\hat{k}^*)>U_i(\frac{1+\delta\cos\alpha}{1+\delta}+\epsilon|\hat{k}^*)$, $\mathbf{p}_i((\frac{1+\delta\cos\alpha}{1+\delta}+\epsilon)\mathbf{b}_i)=(\frac{1+\delta\cos\alpha}{1+\delta}+\epsilon)\mathbf{b}_i$ and $d(\frac{1+\delta\cos\alpha}{1+\delta}\mathbf{b}_i)<d((\frac{1+\delta\cos\alpha}{1+\delta}+\bar{\epsilon})\mathbf{b}_i)$, a contradiction. Finally suppose, towards a third contradiction, that $\hat{k}^*_i>1$. Then irrespective of $\hat{k}^*_{-i}$, Lemma A1 implies that there exists $\bar{\epsilon}>0$ such that, for all $\epsilon\in(0,\bar{\epsilon})$, $U_i'(1+\epsilon|\hat{k}^*)<0$, which implies that $U_i(1|\hat{k}^*)>U_i(1+\epsilon|\hat{k}^*)$, $\mathbf{p}_i((1+\epsilon)\mathbf{b}_i)=(1+\epsilon)\mathbf{b}_i$

[22] For completeness:

$$\begin{aligned}
\chi_1 &= u_i(\hat{k}_i\mathbf{b}_i)\delta^2+r(1+\delta^2)+\delta^2V_i^i(\hat{k}_i\mathbf{b}_i|\hat{k})\\
\chi_2 &= u_i(\hat{k}_{-i}\mathbf{b}_{-i})\delta+u_i(\hat{k}_{-i}\tfrac{b_{-i}}{b_i}\mathbf{b}_i)\delta^2+r(1+\delta^2)+\delta^2V_i^i(\hat{k}_{-i}\tfrac{b_{-i}}{b_i}\mathbf{b}_i|\hat{k})\\
\chi_3 &= u_i(\hat{k}_{-i}\mathbf{b}_{-i})\delta+u_i(\min\{\hat{k}_{-i}\tfrac{b_{-i}}{b_i},\hat{k}_i\}\mathbf{b}_i)\delta^2+r(1+\delta^2)+\delta^2V_i^i(\min\{\hat{k}_{-i}\tfrac{b_{-i}}{b_i},\hat{k}_i\}\mathbf{b}_i|\hat{k})
\end{aligned}$$

and $d(\mathbf{b}_i) < d((1+\bar{\epsilon})\mathbf{b}_i)$, a contradiction.

Having shown that $\hat{k}_i^* \in \{\frac{1+\delta\cos\alpha}{1+\delta}, 1\}$ for $i \in \{1,2\}$, we now argue that $\hat{k}_i^* = \frac{1+\delta\cos\alpha}{1+\delta}$ for all $i \in \{1,2\}$ cannot constitute an SSMPE unless $\frac{1+\delta\cos\alpha}{1+\delta} = 1$, that is unless $\delta = 0$ or $\alpha = 0$. Suppose, towards a contradiction, that $\delta > 0$, $\alpha > 0$ and $\hat{k}_i^* = \frac{1+\delta\cos\alpha}{1+\delta}$. This implies that $\hat{k}_i^* < 1$ for all $i \in \{1,2\}$. Suppose, without loss of generality, that $b_1 \geq b_2$. Then Lemma A1 implies that there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0,\bar{\epsilon})$, $U_1'(\hat{k}_1^* + \epsilon|\hat{k}^*) > 0$, which implies that $U_1(\hat{k}_1^*|\hat{k}^*) < U_1(\hat{k}_1^* + \epsilon|\hat{k}^*)$, $\mathbf{p}_1((\hat{k}_1^*+\epsilon)\mathbf{b}_1) = \hat{k}_1^*\mathbf{b}_1$ and $d((\hat{k}_1^*+\epsilon)\mathbf{b}_1) < d((\hat{k}_1^*+\bar{\epsilon})\mathbf{b}_1)$, which means party 1 is not maximizing its dynamic utility when at the opposition. Furthermore, an argument similar to the one used in the second contradiction above implies that $\hat{k}_1^* = \hat{k}_2^* = 1$ cannot constitute an SSMPE unless $\delta = 0$ or $\alpha = 0$.

This leaves three possible cases. *Case 1:* $\delta = 0$ or $\alpha = 0$ and $\hat{k}_1^* = \hat{k}_2^* = 1$. When $\delta = 0$ or $\alpha = 0$, clearly $\hat{k}^* = (1,1)$ constitutes a SSMPE and, because $\hat{k}_i^* \in \{\frac{1+\delta\cos\alpha}{1+\delta}, 1\}$ for $i \in \{1,2\}$, this SSMPE is unique. *Case 2:* $\delta > 0$ and $\alpha > 0$ and $\hat{k}_1^* = 1$ with $\hat{k}_2^* = \frac{1+\delta\cos\alpha}{1+\delta}$. *Case 3:* $\delta > 0$ and $\alpha > 0$ and $\hat{k}_1^* = \frac{1+\delta\cos\alpha}{1+\delta}$ with $\hat{k}_2^* = 1$.

We first claim that when $b_1 > b_2$, Case 3 cannot constitute a SSMPE. When $\delta > 0$ and $\alpha > 0$, $\frac{1+\delta\cos\alpha}{1+\delta} < 1$. From $\hat{k}_1^* = \frac{1+\delta\cos\alpha}{1+\delta}$, $\hat{k}_2^* = 1$ and $b_1 > b_2$, it follows $\frac{1+\delta\cos\alpha}{1+\delta}b_2 < b_2 = \hat{k}_2^* b_2$ and $\frac{1+\delta\cos\alpha}{1+\delta}b_2 < \frac{1+\delta\cos\alpha}{1+\delta}b_1 = \hat{k}_1^* b_1$, or $\frac{1+\delta\cos\alpha}{1+\delta} < \min\{\hat{k}_1^*\frac{b_1}{b_2}, \hat{k}_2^*\}$. We can now use argument similar to the one used in the second contradiction above to establish contradiction with party 2 maximizing its dynamic utility when in opposition.

It remains to be shown that, when $b_1 > b_2$, Case 2 constitutes a unique SSMPE and that, when $b_1 = b_2$, both Cases 2 and 3 constitute an SSMPE. We cover Case 2 irrespective of whether $b_1 > b_2$ or $b_1 = b_2$. When $b_1 = b_2$, Case 3 is similar to Case 2 and is omitted.

Suppose $\delta > 0$, $\alpha > 0$ and $b_1 \geq b_2$. We need to show that $\hat{k}_1 = 1$ and $\hat{k}_2 = \frac{1+\delta\cos\alpha}{1+\delta}$ constitute an SSMPE. Take any incumbent's policy commitment $\mathbf{q} \in X$. The optimization problem of the opposition party regarding the policy to contest elections with is $\max_{k\in[0,k_i(\mathbf{q})]} U_i(k|\hat{k})$. For any $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) > \hat{k}_2$, from Lemma A1 we have $\lim_{k\to\hat{k}_2^-} U_2'(k|\hat{k}) \geq 0$ and $\lim_{k\to\hat{k}_2^+} U_2'(k|\hat{k}) \leq 0$. By piece-wise strict concavity of U_2 established in the same lemma, $U_2(k|\hat{k})$ is, for any $\mathbf{q} \in X$, increasing in k on $[0, \min\{\hat{k}_2, k_2(\mathbf{q})\}]$ and decreasing in k on $[\hat{k}_2, \max\{\hat{k}_2, k_2(\mathbf{q})\}]$. When $k_2(\mathbf{q}) \leq \hat{k}_2$, it is optimal for party 2 to contest elections with $k_2(\mathbf{q})\mathbf{b}_2$ and when $k_2(\mathbf{q}) > \hat{k}_2$, it is optimal to run with $\hat{k}_2\mathbf{b}_2$. The simple strategy with $\hat{k}_2$ is thus optimal for party 2. A similar argument can be used to show optimality of the simple strategy with $\hat{k}_1$ for party 1. The key to this claim is Lemma A1 along with $\hat{k}_2 b_2 < \hat{k}_1 b_1$ and $\hat{k}_2 b_2 = \frac{1+\delta \cos \alpha}{1+\delta} b_2 < \frac{1+\delta \cos \alpha}{1+\delta} b_1$.

To finish the proof of parts 1 and 2 of the proposition, we have to show that for none of the parties, when at the opposition, staying out of the election is a profitable deviation. Take any $\hat{k}^*$ characterized in the three cases above and any incumbent's policy commitment $\mathbf{q} \in X$. The dynamic utility of opposition party $i \in \{1, 2\}$ on the equilibrium path is $u_i(\mathbf{p}_i(\mathbf{q})) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{q})|\hat{k}^*)$ whereas the dynamic utility from staying out is $u_i(\mathbf{q}) + \delta V_i^{-i}(\mathbf{q}|\hat{k}^*)$. Because on the equilibrium path i contests elections, $V_i^{-i}(\mathbf{q}|\hat{k}^*) = u_i(\mathbf{p}_i(\mathbf{q})) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{q})|\hat{k}^*)$. We need to ensure that the on-path dynamic utility is larger than the off-path one, or $u_i(\mathbf{p}_i(\mathbf{q})) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{q})|\hat{k}^*) \geq \frac{u_i(\mathbf{q})}{1-\delta}$. We derive an upper bound on the right hand side of the inequality and a lower bound on the left hand side of the inequality and show that the upper bound is smaller than the lower bound.

The upper bound is clearly $0 \geq \frac{u_i(\mathbf{q})}{1-\delta}$. We construct the lower bound as follows. $u_i(\mathbf{p}_i(\mathbf{q})) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{q})|\hat{k}^*)$ is the dynamic utility of opposition party i from running with the optimal policy, given incumbent's policy commitment $\mathbf{q} \in X$. Because $\mathbf{p}_i$ is a simple strategy, $U_i(\min\{k_i(\mathbf{q}), \hat{k}_i^*\}|\hat{k}^*) = u_i(\mathbf{p}_i(\mathbf{q})) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{q})|\hat{k}^*)$. From Lemma A1 and the discussion that followed, $U_i(k|\hat{k}^*)$ is increasing in k on $[0, \hat{k}_i^*]$ in any SSMPE. Hence $U_i(0|\hat{k}^*) \leq u_i(\mathbf{p}_i(\mathbf{q})) + r + \delta V_i^i(\mathbf{p}_i(\mathbf{q})|\hat{k}^*)$ for any $\mathbf{q} \in X$. From the proof of Lemma A1, $U_i(0|\hat{k}^*) = \frac{-d^2(0,\mathbf{b}_i)(1+\delta)+r}{1-\delta^2}$. $\frac{-d^2(0,\mathbf{b}_i)(1+\delta)+r}{1-\delta^2} \geq 0$ or equivalently $r \geq d^2(0, \mathbf{b}_i)(1 + \delta)$ then ensures that none of the parties, when at the opposition, wants to stay out of the elections. This concludes the proof of parts 1 and 2.

To prove part 4 of the proposition, take the SSMPE $\hat{k}_1^* = 1$ and $\hat{k}_2^* = \frac{1+\delta \cos \alpha}{1+\delta}$ and $b_1 \geq b_2$, so that $\hat{k}_1^* b_1 \geq \hat{k}_2^* b_2$. Starting from $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) \leq \hat{k}_2^*$, $k_2(\mathbf{q}) b_2 = k_1(\mathbf{q}) b_1 \leq \hat{k}_2^* b_2 \leq \hat{k}_1^* b_1$, both parties run with $k_i(\mathbf{q})\mathbf{b}_i$ when at the opposition in the first period. Since $d(k_i(\mathbf{q})\mathbf{b}_i) = d(\mathbf{q})$, the incumbent's policy commitment at the beginning of the second period is a policy at the same distance from the origin as $\mathbf{q}$. Thus both parties run with $k_i(\mathbf{q})\mathbf{b}_i$ when at the opposition in the second period. The same holds in any future period. Starting from $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) \geq \hat{k}_2^*$, if party 2 is at the opposition in the first period it runs with $\hat{k}_2^* \mathbf{b}_2$ and if party 1 is at the opposition it runs with $\min\{k_1(\mathbf{q}), \hat{k}_1^*\}\mathbf{b}_1$. In the former case, the incumbent's policy commitment at the beginning of the second period satisfies $k_2(\hat{k}_2^* \mathbf{b}_2) = \hat{k}_2^*$, so that the policies alternate on $k_2(\hat{k}_2^* \mathbf{b}_2)\mathbf{b}_2 = \hat{k}_2^* \mathbf{b}_2$ and $k_1(\hat{k}_2^* \mathbf{b}_2)\mathbf{b}_1$ starting from period 2. In the latter case, because $k_2(\min\{k_1(\mathbf{q}), \hat{k}_1^*\}\mathbf{b}_1) = \min\{k_1(\mathbf{q}), \hat{k}_1^*\}\frac{b_1}{b_2} =$

$\min\{k_2(\mathbf{q}), \hat{k}_1^* \frac{b_1}{b_2}\} \geq \hat{k}_2^*$, party 2 in the second period runs with $\hat{k}_2^* \mathbf{b}_2$ and the same alternation obtains from period 3 onwards. $\square$

Proof of Lemma 1

In the SSMPE from Proposition 1, the opposition party $i \in \{1, 2\}$ contests elections and wins with policy $\mathbf{p}_i(\mathbf{q})$, for any incumbent's policy commitment $\mathbf{q} \in X$. Because $u_v(\mathbf{x}) = -d^2(\mathbf{x})$ is decreasing in $d(\mathbf{x})$ for any $\mathbf{x} \in X$ and $d(\mathbf{p}_i(\mathbf{q})) = \min\{k_i(\mathbf{q}), \hat{k}_i\} b_i$ for any $\mathbf{q} \in X$, all we have to show is that $\min\{k_i(\mathbf{q}), \hat{k}_i\} b_i$ is non-decreasing in $\hat{k}_2$, b_1 and b_2 for all $i \in \{1, 2\}$ and $\mathbf{q} \in X$ and is increasing in $\hat{k}_2$, b_1 and b_2 for some $i \in \{1, 2\}$ and $\mathbf{q} \in X$. The non-decreasing part is immediate. For the increasing part, we have $\min\{k_1(\mathbf{q}), \hat{k}_1\} b_1 = \hat{k}_1 b_1$ when $k_1(\mathbf{q}) > \hat{k}_1$, which is increasing in b_1 because $\hat{k}_1 = 1$, and $\min\{k_2(\mathbf{q}), \hat{k}_2\} b_2 = \hat{k}_2 b_2$ when $k_2(\mathbf{q}) > \hat{k}_2$, which is increasing in $\hat{k}_2$ and b_2 because $b_2 > 0$ and $\hat{k}_2 = \frac{1+\delta\cos\alpha}{1+\delta} > 0$ for any $\alpha \in [0, \pi]$ and $\delta \in [0, 1)$. $\square$

Proof of Proposition 3

By Proposition 1, $\hat{k}_1 = 1$ and $\hat{k}_2 = \frac{1+\delta\cos\alpha}{1+\delta}$. Notice that $\hat{k}_2 \in (0, 1]$ for any $\delta \in [0, 1)$ and $\alpha \in [0, \pi]$. To show part 1, using the transformation of polar coordinated to Cartesian ones — $\hat{k}_2 \mathbf{b}_2 = (\hat{k}_2 b_2 \cos\alpha, \hat{k}_2 b_2 \sin\alpha)$ — d_p can be expressed as $d_p = 2b_2\hat{k}_2 \sin\frac{\alpha}{2}$. d_p is clearly non-decreasing in b_2, increasing in b_2 when $\alpha > 0$ and constant in b_1. To show part 2,

$$\frac{\partial\left[2b_2\frac{1+\delta\cos\alpha}{1+\delta}\sin\frac{\alpha}{2}\right]}{\partial\alpha} = \frac{b_2\cos\frac{\alpha}{2}}{1+\delta}[1 - 2\delta + 3\delta\cos\alpha] \tag{A7}$$

where the first term is positive unless $\alpha = \pi$. Solving $1 - 2\delta + 3\delta\cos\alpha = 0$ gives $\alpha = \arccos\left[\frac{2\delta-1}{3\delta}\right]$. From $\{x | \arccos x \in \mathbb{R}\} = [-1, 1]$, $-1 \leq \frac{2\delta-1}{3\delta} \leq 1$ holds when $\delta \geq \frac{1}{5}$. Because $\arccos x \in [0, \pi)$ for $x \in (-1, 1]$, $\arccos\left[\frac{2\delta-1}{3\delta}\right] < \pi$ when $\delta > \frac{1}{5}$. Defining $\alpha' = \arccos\left[\frac{2\delta-1}{3\delta}\right]$ when $\delta \geq \frac{1}{5}$ and $\alpha' = \pi$ when $\delta \in [0, \frac{1}{5})$, $\alpha' < \pi$ when $\delta > \frac{1}{5}$. That d_p is increasing in α when $\alpha \in [0, \alpha']$ and decreasing in α when $\alpha \in [\alpha', \pi]$ then follows from the fact that $\cos\alpha$, and hence $1 - 2\delta + 3\delta\cos\alpha$, is decreasing in α. $\square$

A2 Extensions

A2.1 Model with General Utility Functions

The model analyzed in this section is identical to the one in the main part of the paper except for the stage utility player $i \in \{1, 2, v\}$ derives from policy $\mathbf{p}$ which now is

$$u_i(\mathbf{p}) = f(d(\mathbf{p}, \mathbf{b}_i)) \tag{A8}$$

where $f : [0, \infty) \to \mathbb{R}$ is a continuous, decreasing and concave function in $d(\mathbf{p}, \mathbf{b}_i)$. We also assume that f is twice continuously differentiable on $(0, \infty)$. Notice that these assumptions allow $f'(0) = 0$ and $f(x) = -x^2$, so that the model in the paper is a special case of the model analyzed here.

For space considerations, we refrain from repeating the arguments leading to Proposition 1 and stress only those aspects of the analysis that differ considerably. An argument similar to the one from the proof of Proposition 1 shows that, since parties use simple strategies, it is i) optimal for $i \in \{1, 2\}$ to contest elections with policies located only on the $\mathbf{b}_i$-ray and ii) optimal for v to vote for the opposition party running with $\mathbf{p}$ when the incumbent's policy commitment is $\mathbf{q}$ if and only if $d(\mathbf{p}) \leq d(\mathbf{q})$.

In an SSMPE, the extent of moderation of party $i \in \{1, 2\}$ is the k which maximizes $\tilde{U}_i(k)$ where, for $k \geq 0$,

$$\begin{aligned} \tilde{U}_i(k) &= u_i(k\mathbf{b}_i) + u_i(kb_{ir}\mathbf{b}_{-i})\delta \\ &= f(d(k\mathbf{b}_i, \mathbf{b}_i)) + f(d(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i))\delta. \end{aligned} \tag{A9}$$

$\tilde{U}_i(k)$ is the dynamic utility party i receives, devoid of any constant terms, when it runs with policy $k\mathbf{b}_i$ and party $-i$, not moderating to a larger extent, runs with $kb_{ir}\mathbf{b}_{-i}$, where we are using the shorthand $b_{ir} = \frac{b_i}{b_{-i}}$. The derivation of $\tilde{U}_i(k)$ is similar to the derivation of (A6) and is not repeated here.

Condition A1 (Non trivial model)**.** *$\delta \in (0, 1)$ and $\alpha \neq 0$.*

Condition A2 (Preserving concavity)**.** *If $\alpha = \pi$, then $f'' < 0$.*[23]

Condition A3 (Concavity at origin)**.** *$f'(0) > f'(b_i\sqrt{2(1-\cos\alpha)})\delta \sin\frac{\alpha}{2}$ for $i \in \{1, 2\}$.*

[23] With a slight abuse of terminology, when Condition A2 fails we mean $\alpha = \pi$ and $f'' = 0$, that is, f is linear. Formally, the failure of $f'' < 0$ permits an f that is linear in some parts of its domain and strictly concave in others. Characterizing these intermediate cases provides little additional insights.

Lemma A2. *$\tilde{U}_i(k)$ is*

1. *continuous and twice continuously differentiable for $k \neq 1$*
2. *increasing for $k \in [0, \cos\alpha]$*
3. *decreasing for $k \geq 1$*
4. *if Conditions A1 and A2 hold, strictly concave for $k \in (\max\{0, \cos\alpha\}, 1)$*
5. *if Condition A1 fails, increasing for $k \in (\max\{0, \cos\alpha\}, 1)$*
6. *if Condition A2 fails, increasing for $k \in (0, 1)$*

Proof. The continuity in part 1 follows from the continuity of f, $d(k\mathbf{b}_i, \mathbf{b}_i)$ and $d(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)$. The differentiability follows from the derivatives below, which can be easily checked

$$d(k\mathbf{b}_i, \mathbf{b}_i) = b_i|1-k| \qquad d(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i) = b_i\sqrt{(k-\cos\alpha)^2 + \sin^2\alpha}$$

$$d'(k\mathbf{b}_i, \mathbf{b}_i) \begin{cases} < 0 & \text{if } k \in [0,1) \\ \nexists & \text{if } k = 1 \\ > 0 & \text{if } k \in (1, \infty) \end{cases} \qquad d'(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i) \begin{cases} < 0 & \text{if } k < \cos\alpha \\ = 0 & \text{if } k = \cos\alpha \wedge \alpha \neq 0 \\ \nexists & \text{if } k = \cos\alpha \wedge \alpha = 0 \\ > 0 & \text{if } k > \cos\alpha \end{cases}$$

$$d''(k\mathbf{b}_i, \mathbf{b}_i) \begin{cases} = 0 & \text{if } k \neq 1 \\ \nexists & \text{if } k = 1 \end{cases} \qquad d''(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i) \begin{cases} > 0 & \text{if } \alpha \notin \{0, \pi\} \\ 0 & \text{if } \begin{array}{l} \alpha = \pi \vee \\ (k \neq 1 \wedge \alpha = 0) \end{array} \\ \nexists & \text{if } k = 1 \wedge \alpha = 0. \end{cases}$$

For parts 2 and 3, we have

$$\tilde{U}_i'(k) = f'(d(k\mathbf{b}_i, \mathbf{b}_i))d'(k\mathbf{b}_i, \mathbf{b}_i) + f'(d(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i))d'(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)\delta \tag{A10}$$

and direct verification shows that $\tilde{U}_i'(k) > 0$ for $k < \cos\alpha$, $\lim_{k\to\cos\alpha^-} \tilde{U}_i'(k) \geq 0$, $\tilde{U}_i'(k) < 0$ for $k > 1$ and $\lim_{k\to 1^+} \tilde{U}_i'(k) \leq 0$.

For the strict concavity for $k \in (\max\{0, \cos\alpha\}, 1)$ in part 4, we have

$$\begin{aligned} \tilde{U}_i''(k) = \quad & f''(d(k\mathbf{b}_i, \mathbf{b}_i))\left[d'(k\mathbf{b}_i, \mathbf{b}_i)\right]^2 + f'(d(k\mathbf{b}_i, \mathbf{b}_i))d''(k\mathbf{b}_i, \mathbf{b}_i) + \\ & \delta f''(d(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i))\left[d'(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)\right]^2 + f'(d(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i))d''(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)\delta. \end{aligned} \tag{A11}$$

$\tilde{U}_i''(k) \leq 0$ is a consequence of the fact that all the summands of the expression are non-positive. To see that $\tilde{U}_i''(k) < 0$ under Conditions A1 and A2, note that the last summand is either zero (when $d''(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i) = 0 \Rightarrow \alpha = \pi$), which implies that the next to last summand is negative ($\alpha = \pi \Rightarrow f'' < 0$ by Condition A2), or negative.

For part 5, which claims that $\tilde{U}_i(k)$ is increasing on $(\max\{0, \cos\alpha\}, 1)$ when Condition A1 fails, notice that the failure of the condition implies either $\delta = 0$ or $\alpha = 0$. In the latter case the $(\max\{0, \cos\alpha\}, 1)$ interval is empty so assume $\delta = 0$ and $\alpha > 0$. Substituting $\delta = 0$ into (A9) gives $\tilde{U}_i(k) = f(d(k\mathbf{b}_i, \mathbf{b}_i))$ so that $\tilde{U}_i(k)$ is increasing on $(\max\{0, \cos\alpha\}, 1)$. Finally, when Condition A2 in part 6 fails, we have $\alpha = \pi$ and $f'' = 0$ so that $d'(k\mathbf{b}_i, \mathbf{b}_i) = -d'(kb_{ir}\mathbf{b}_{-i}, \mathbf{b}_i) = -b_i$ for $k \in (0, 1)$. This implies that $\tilde{U}_i'(k) = (1 - \delta)c$ where $c > 0$ so that $\tilde{U}_i(k)$ is increasing on $(0, 1)$. □

Proposition A1 (SSMPE with General Utility)**.** *When the stage utility of $i \in \{1, 2, v\}$ is $u_i(\mathbf{p}) = f(d(\mathbf{p}, \mathbf{b}_i))$ where f is a twice continuously differentiable, decreasing and concave function, then:*

1. *if Condition A1 fails, $\hat{k}_1 = \hat{k}_2 = 1$ characterize the unique SSMPE;*
2. *if Condition A2 fails, $\hat{k}_1 = \hat{k}_2 = 1$ characterize the unique SSMPE;*
3. *if Conditions A1 and A2 hold, there exists a unique κ_i for $i \in \{1, 2\}$ given by $\kappa_i = \arg\max_{k\geq 0} \tilde{U}_i(k) \in [\max\{0, \cos\alpha\}, 1]$ and either $\kappa_i b_i > \kappa_{-i} b_{-i}$, in which case $\hat{k}_i = 1$ and $\hat{k}_{-i} = \kappa_{-i}$ characterize the unique SSMPE, or $\kappa_1 b_1 = \kappa_2 b_2$, in which case there exist exactly two SSMPE characterized by $\hat{k}_1 = 1$, $\hat{k}_2 = \kappa_2$ and $\hat{k}_1 = \kappa_1$, $\hat{k}_2 = 1$.*

Proof. When Condition A1 fails, either $\delta = 0$ or $\alpha = 0$, so that none of the players has any incentive to moderate. By Lemma A2 there exists a unique maximizer of $\tilde{U}_i(k)$ for $i \in \{1, 2\}$, $k = 1$, and using arguments similar to the proof of Proposition 1, there exists a unique SSMPE characterized in part 1 of the proposition. When Condition A2 fails, then by Lemma A2 the maximizer of $\tilde{U}_i(k)$ is $k = 1$ for $i \in \{1, 2\}$. Repeating the same argument just made, there exists a unique SSMPE characterized in part 2 of the proposition.

When both Conditions A1 and A2 hold in part 3, the uniqueness of $\kappa_i = \arg\max_{k\geq 0} \tilde{U}_i(k)$ and $\kappa_i \in [\max\{0, \cos\alpha\}, 1]$ follows from Lemma A2. Recalling again the proof of Proposition 1, uniqueness/multiplicity of a SSMPE and its characterization follows.[24] □

[24] With general utility it is not necessarily true that $b_1 > b_2$ implies $\kappa_1 b_1 > \kappa_2 b_2$, even though

counter-examples are difficult to produce. Using similar argument as in the proof of Proposition A2 below, a sufficient condition for $b_1 > b_2 \Rightarrow \kappa_1 b_1 > \kappa_2 b_2$ is $\frac{\partial}{\partial b_i}\tilde{U}_i'(\kappa_i) \geq 0$. This condition, for strictly concave f, rewrites as $\kappa_i \leq \frac{z+\delta\cos\alpha}{z+\delta}$ where $z = \frac{f''(b_i(1-\kappa_i))}{f''(b_i\sqrt{(\kappa_i-\cos\alpha)^2+\sin^2\alpha})} > 0$ and holds for high z or low δ and α. The counter-example we were able to produce uses high δ and α along with $f(x) = -\exp x^2$.

Proposition A2 (SSMPE Comparative Static with General Utility)**.**
Assume Conditions A1, A2 and A3 hold. Then $\kappa_i \in (\max\{0,\cos\alpha\},1)$ *for* $i \in \{1,2\}$ *in Proposition A1 and (assuming* $\alpha \neq \pi$ *for the first relation)*

$$\frac{\partial \kappa_i}{\partial \alpha} < 0 \qquad \frac{\partial \kappa_i}{\partial \delta} < 0 \tag{A12}$$

Proof. From Proposition A1, if Conditions A1 and A2 hold, $\kappa_i = \arg\max_{k\geq 0}\tilde{U}_i(k) \in [\max\{0,\cos\alpha\},1]$ for $i \in \{1,2\}$. We need to show that Condition A3 implies $\kappa_i \in (\max\{0,\cos\alpha\},1)$ for $i \in \{1,2\}$. We show that Condition A3 is required only for $\kappa_i < 1$ and that $\kappa_i > \max\{0,\cos\alpha\}$ holds in general. Fix $i \in \{1,2\}$. By Lemma A2 part 4, $\tilde{U}_i(k)$ is strictly concave for $k \in (\max\{0,\cos\alpha\},1)$. To show that $\kappa_i < 1$, it thus suffices to show that $\lim_{k\to 1^-}\tilde{U}_i'(k) < 0$. Substituting

$$\begin{aligned} &\lim_{k\to 1^-} d(k\mathbf{b}_i,\mathbf{b}_i)) = 0 && \lim_{k\to 1^-} d'(k\mathbf{b}_i,\mathbf{b}_i)) = -b_i \\ &\lim_{k\to 1^-} d(kb_{ir}\mathbf{b}_{-i},\mathbf{b}_i)) = b_i\sqrt{2(1-\cos\alpha)} && \lim_{k\to 1^-} d'(kb_{ir}\mathbf{b}_{-i},\mathbf{b}_i)) = b_i\sin\tfrac{\alpha}{2} \end{aligned} \tag{A13}$$

into (A10) gives $\lim_{k\to 1^-}\tilde{U}_i'(k) = -b_i[f'(0) - f'(b_i\sqrt{2(1-\cos\alpha)})\delta\sin\frac{\alpha}{2}] < 0$, where the inequality follows from Condition A3. To show that $\kappa_i > \max\{0,\cos\alpha\}$, it suffices to show that $\lim_{k\to 0^+}\tilde{U}_i'(k) > 0$ when $\alpha \geq \frac{\pi}{2}$ and $\lim_{k\to\cos\alpha^+}\tilde{U}_i'(k) > 0$ when $\alpha < \frac{\pi}{2}$. When $\alpha \geq \frac{\pi}{2}$, substituting

$$\begin{aligned} &\lim_{k\to 0^+} d(k\mathbf{b}_i,\mathbf{b}_i)) = b_i && \lim_{k\to 0^+} d'(k\mathbf{b}_i,\mathbf{b}_i)) = -b_i \\ &\lim_{k\to 0^+} d(kb_{ir}\mathbf{b}_{-i},\mathbf{b}_i)) = b_i && \lim_{k\to 0^+} d'(kb_{ir}\mathbf{b}_{-i},\mathbf{b}_i)) = -b_i\cos\alpha \end{aligned} \tag{A14}$$

into (A10) gives $\lim_{k\to 0^+}\tilde{U}_i'(k) = -b_i f'(b_i)(1+\delta\cos\alpha) > 0$. When $\alpha < \frac{\pi}{2}$, substituting

$$\begin{aligned} &\lim_{k\to \cos\alpha^+} d(k\mathbf{b}_i,\mathbf{b}_i)) = b_i(1-\cos\alpha) && \lim_{k\to \cos\alpha^+} d'(k\mathbf{b}_i,\mathbf{b}_i)) = -b_i \\ &\lim_{k\to \cos\alpha^+} d(kb_{ir}\mathbf{b}_{-i},\mathbf{b}_i)) = b_i\sin\alpha && \lim_{k\to \cos\alpha^+} d'(kb_{ir}\mathbf{b}_{-i},\mathbf{b}_i)) = 0 \end{aligned} \tag{A15}$$

into (A10) gives $\lim_{k\to\cos\alpha^+}\tilde{U}_i'(k) = -b_i f'(b_i(1-\cos\alpha)) > 0$.

Since $\kappa_i \in (\max\{0,\cos\alpha\},1)$ under the conditions of the proposition, it is implicitly defined by $\tilde{U}_i'(\kappa_i) = 0$. From the implicit function theorem $\frac{\partial \kappa_i}{\partial x} = \frac{\frac{\partial}{\partial x}\tilde{U}_i'(\kappa_i)}{-\tilde{U}_i''(\kappa_i)}$. The denominator of this expression is positive by strict concavity of $\tilde{U}_i(k)$. The numerator of this expression for $x \in \{\alpha, \delta\}$ is

$$\begin{aligned}
\tfrac{\partial}{\partial \alpha}\tilde{U}_i'(\kappa_i) &= f''(d(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i))d'(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)\frac{\partial d(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)}{\partial \alpha}\delta \\
&\quad + f'(d(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i))\frac{\partial d'(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)}{\partial \alpha}\delta \\
\tfrac{\partial}{\partial \delta}\tilde{U}_i'(\kappa_i) &= f'(d(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i))d'(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i)
\end{aligned} \tag{A16}$$

where both expressions are negative since $\frac{\partial}{\partial \alpha}d(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i) = \frac{b_i \kappa_i \sin\alpha}{\sqrt{(\kappa_i - \cos\alpha)^2 + \sin^2\alpha}} > 0$ and $\frac{\partial}{\partial \alpha}d'(\kappa_i b_{ir}\mathbf{b}_{-i}, \mathbf{b}_i) = \frac{b_i \sin\alpha(1 - \kappa_i\cos\alpha)}{\left(\sqrt{(\kappa_i - \cos\alpha)^2 + \sin^2\alpha}\right)^3} > 0$.[25] $\square$

A2.2 Forward Looking Representative Voter

In this section we study a version of the model from the main part of the paper in which the representative voter is forward-looking. We will show that for any SSMPE identified in Proposition 1, there exists an equilibrium with forward-looking v and that this equilibrium generates a comparative static on the representative voter's welfare with respect to antagonism and extremism which is identical to the one stated in Corollary 1.

Throughout this section assume $\delta_v \in [0, 1)$. The model from the main part requires the following changes. The utility v derives from a sequence of policies $\mathbf{P} = \{\mathbf{p}_0, \mathbf{p}_1, \ldots\}$ is the discounted sum of payoffs from each period

$$U_v(\mathbf{P}) = \sum_{t=0}^{\infty} \delta_v^t u_v(\mathbf{p}_t). \tag{A17}$$

Given a path of policies generated by play according to $\sigma = (\sigma_1, \sigma_2, \sigma_v)$ starting from an incumbent $j \in \{1, 2\}$ committed to $\mathbf{q}$, $\mathbf{P}(j, \sigma, \mathbf{q}) = \{\mathbf{p}_0, \mathbf{p}_1, \ldots\}$, the dynamic utility of v is

$$V_v^j(\mathbf{q}|\sigma) = U_v(\mathbf{P}(j, \sigma, \mathbf{q})) = \sum_{t=0}^{\infty} \delta_v^t u_v(\mathbf{p}_t). \tag{A18}$$

[25] Sufficient condition for $\kappa_i b_i$ to be increasing in b_i, for extremism to be welfare reducing, is $\frac{\partial}{\partial b_i}\tilde{U}_i'(\kappa_i) \geq 0$, which is, for strictly concave f, guaranteed if $\frac{x f''(x)}{y f''(y)} \geq \frac{f'(x)}{f'(y)}$ for any $y > x > 0$. When this condition holds with equality, which is true for any power function, κ_i does not depend on b_i.

In Definition 2 of SMPE, we now require $\sigma_v^*(j, \mathbf{q}, \mathbf{p}) = Yes$ if and only if

$$u_v(\mathbf{p}) + \delta_v V_v^i(\mathbf{p}|\sigma^*) \geq u_v(\mathbf{q}) + \delta_v V_v^j(\mathbf{q}|\sigma^*) \tag{A19}$$

for any incumbent $j \in \{1, 2\}$ committed to $\mathbf{q} \in X$ and the opposition $i = \{1, 2\}\setminus\{j\}$ contesting elections with $\mathbf{p} \in X$. Given a profile of strategies σ, assuming the initial incumbent's policy commitment $\mathbf{q}^0$ is distributed according to $F(\mathbf{q}^0)$ with strictly positive density on X, and denoting with $\rho \in (0, 1)$ the probability that party 1 is the first-period incumbent, the representative voter's ex-ante welfare from the dynamic electoral competition game with parties' extremism b_1 and b_2 and antagonism α is

$$W(b_1, b_2, \alpha|\sigma) = \int_X \rho V_v^1(\mathbf{z}|\sigma) + (1 - \rho)V_v^2(\mathbf{z}|\sigma)d(F(\mathbf{z})). \tag{A20}$$

Proposition A3. *For any SSMPE σ' in the model with $\delta_v = 0$ identified in Proposition 1, there exists a SMPE σ^* in the model with $\delta_v \in (0, 1)$ such that:*

1. *in σ^*, elections are always contested and incumbents always defeated;*
2. *starting from $\mathbf{q} \in X$, policies converge to identical alternation under σ' and σ^*;*
3. *$W(b_1, b_2, \alpha|\sigma^*)$ is non-decreasing in α; increasing in α if $\delta > 0$; decreasing in b_i for $i \in \{1, 2\}$.*

Proof. Fix the SSMPE $\sigma' = (\sigma_1', \sigma_2', \sigma_v')$ from Proposition 1. We construct an SMPE $\sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_v^*)$ for the model with $\delta_v \in [0, 1)$ and then show that it possesses the properties from parts 1 through 3. Assume that σ' is such that party 2 moderates, that is $\sigma_1'(\mathbf{q}) = \mathbf{p}_1(\mathbf{q})$ with $\hat{k}_1 = 1$ and $\sigma_2'(\mathbf{q}) = \mathbf{p}_2(\mathbf{q})$ with $\hat{k}_2 = \frac{1+\delta\cos\alpha}{1+\delta}$ for any $\mathbf{q} \in X$. For the 'mirror' SSMPE identified in Proposition 1 part 2, the proof is similar and omitted.[26]

[26] The reason why σ' is not an SMPE when $\delta_v > 0$ is the fact that rejecting $\mathbf{p}$ of the opposition party and voting for incumbent's policy commitment $\mathbf{q}$ whenever $d(\mathbf{p}) > d(\mathbf{q})$ need not be optimal for v. To see this suppose the moderating party 2 is committed to $\mathbf{q} \in X$ with $d(\mathbf{q}) > \hat{k}_2 b_2$. According to σ_1', party 1 contests elections with $\mathbf{p} \in X$ such that $d(\mathbf{p}) = d(\mathbf{q})$. Suppose party 1 deviates and runs with $\mathbf{p}' \in X$ such that $d(\mathbf{p}') = d(\mathbf{q}) + \epsilon$ for some small $\epsilon > 0$. The dynamic utility of v from rejecting $\mathbf{p}'$ is $u_v(\mathbf{q}) + \delta_v u_v(\mathbf{p}) + \delta_v^2 \frac{u_v(\hat{k}_2 \mathbf{b}_2)}{1-\delta_v}$ while the dynamic utility from accepting $\mathbf{p}'$ is $u_v(\mathbf{p}') + \delta_v \frac{u_v(\hat{k}_2 \mathbf{b}_2)}{1-\delta_v}$. The condition for rejection to be optimal rewrites as $u_v(\mathbf{p}) - u_v(\mathbf{p}') > \delta_v(u_v(\hat{k}_2 \mathbf{b}_2) - u_v(\mathbf{p}))$. The left hand side of the condition is positive for any $\epsilon > 0$ and tends to zero as $\epsilon \to 0$. Because the right hand side of the condition is positive whenever $\delta_v > 0$, there exists ϵ small enough such that the condition fails. Intuitively, when the moderating party 2 is committed to $\mathbf{q}$ such that $d(\mathbf{q}) > \hat{k}_2 b_2$, party 1, by contesting elections, releases party 2 from its policy commitment and starts the process of convergence to policies at distance $\hat{k}_2 b_2$ from the

Denote $\tilde{k}_2 = \sqrt{\left(\frac{b_1}{b_2}\right)^2 (1-\delta_v) + \delta_v \hat{k}_2^2}$ and note that, because $\hat{k}_2 \leq 1$ and $\frac{b_1}{b_2} \geq 1$, $\hat{k}_2 \leq \tilde{k}_2 \leq \frac{b_1}{b_2}$. For any $\mathbf{q} \in X$ define

$$\tilde{p}_1(\mathbf{q}) = \begin{cases} k_1(\mathbf{q}) & \text{for} \quad k_2(\mathbf{q}) \leq \hat{k}_2 \\ k_1(\mathbf{q})\sqrt{\frac{1-\delta_v\left(\frac{\hat{k}_2 b_2}{d(\mathbf{q})}\right)^2}{1-\delta_v}} & \text{for} \quad k_2(\mathbf{q}) \in (\hat{k}_2, \tilde{k}_2) \\ 1 & \text{for} \quad k_2(\mathbf{q}) \geq \tilde{k}_2. \end{cases} \tag{A21}$$

Standard arguments show that $\tilde{p}_1(\mathbf{x}) = \tilde{p}_1(\mathbf{y})$ for any $\mathbf{x} \in X$ and $\mathbf{y} \in X$ if $d(\mathbf{x}) = d(\mathbf{y})$ and that $\tilde{p}_1(\mathbf{x})$ is continuous in $d(\mathbf{x})$ and increasing in $d(\mathbf{x})$ if $k_2(\mathbf{x}) \in [0, \tilde{k}_2]$.

We now construct σ^*. For party 2 set $\sigma_2^* = \sigma_2'$, that is σ_2^* is a simple strategy $\mathbf{p}_2$ from Definition 3 with $\hat{k}_2 = \frac{1+\delta\cos\alpha}{1+\delta}$. For party 1 and any $\mathbf{q} \in X$, set $\sigma_1^*(\mathbf{q}) = \tilde{p}_1(\mathbf{q})\mathbf{b}_1$ and note that $\sigma_1^* = \sigma_1'$ when $\delta_v = 0$. For v set σ_v^* such that $\sigma_v^*(j, \mathbf{q}, \mathbf{p}) = Yes$ if and only if, when $j = 1$

$$\begin{aligned} & d(\mathbf{p}) \leq d(\mathbf{q}) && \text{for} \quad k_2(\mathbf{q}) \leq \hat{k}_2 \\ & d(\mathbf{p}) \leq \sqrt{d^2(\mathbf{q})(1-\delta_v) + \delta_v(\hat{k}_2 b_2)^2} && \text{for} \quad k_2(\mathbf{q}) \in (\hat{k}_2, \tfrac{b_1}{b_2}) \\ & d(\mathbf{p}) \leq \sqrt{d^2(\mathbf{q}) - \delta_v b_1^2 + \delta_v(\hat{k}_2 b_2)^2} && \text{for} \quad k_2(\mathbf{q}) \geq \tfrac{b_1}{b_2} \end{aligned} \tag{A22}$$

and when $j = 2$

$$\begin{aligned} & d(\mathbf{p}) \leq d(\mathbf{q}) && \text{for} \quad k_2(\mathbf{q}) \leq \hat{k}_2 \\ & d(\mathbf{p}) \leq \tilde{p}_1(\mathbf{q}) b_1 && \text{for} \quad k_2(\mathbf{q}) \in (\hat{k}_2, \tilde{k}_2) \\ & d(\mathbf{p}) \leq \sqrt{d^2(\mathbf{q}) + \delta_v b_1^2 - \delta_v(\hat{k}_2 b_2)^2} && \text{for} \quad k_2(\mathbf{q}) \geq \tilde{k}_2 \end{aligned} \tag{A23}$$

and note that $\sigma_v^* = \sigma_v'$ when $\delta_v = 0$.

We now argue that $\sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_v^*)$ constitutes an SMPE. From (A22) and $\sigma_2^*(\mathbf{q}) = \min\{k_2(\mathbf{q}), \hat{k}_2\}\mathbf{b}_2$, party 2 contests and wins elections for any policy commitment $\mathbf{q} \in X$ of party 1. From (A23) and $\sigma_1^*(\mathbf{q}) = \tilde{p}_1(\mathbf{q})\mathbf{b}_1$, party 1 contests and wins elections for any policy commitment $\mathbf{q} \in X$ of party 2. Because elections are always contested and incumbents always defeated in σ' as well as in σ^*, because $\sigma_2^*(\mathbf{q}) = \sigma_2'(\mathbf{q})$ and $k_2(\sigma_2^*(\mathbf{q})) \leq \hat{k}_2$ for any $\mathbf{q} \in X$ and because $\sigma_1^*(\mathbf{q}) = \sigma_1'(\mathbf{q})$ for any $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) \leq \hat{k}_2$, it follows that $V_1^1(\mathbf{q}|\sigma^*) = V_1^1(\mathbf{q}|\sigma')$ for any $\mathbf{q} \in X$ and $V_2^2(\mathbf{q}|\sigma^*) = V_2^2(\mathbf{q}|\sigma')$ for any $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) \leq \hat{k}_2$. In addition, because

origin. Because $d(\mathbf{q}) > \hat{k}_2 b_2$, this provides v with a discrete increase in her future utility so that v is willing to accept a moderate decrease in her current utility.

$\tilde{p}_1(\mathbf{q}) \geq k_1(\mathbf{q})$ for any $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) \leq \frac{b_1}{b_2}$ and because $\tilde{p}_1(\mathbf{q}) = \hat{k}_1 = 1$ for any $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) \geq \frac{b_1}{b_2}$, it follows that $V_2^2(\mathbf{q}|\sigma^*) \leq V_2^2(\mathbf{q}|\sigma')$ for any $\mathbf{q} \in X$ such that $k_2(\mathbf{q}) > \hat{k}_2$.

From the proof of Proposition 1 we know that $u_1(k\mathbf{b}_1)+\delta V_1^1(k\mathbf{b}_1|\sigma')$ is increasing in k for $k \in [0,1]$ and decreasing in k for $k \geq 1$. Since $V_1^1(k\mathbf{b}_1|\sigma^*) = V_1^1(k\mathbf{b}_1|\sigma')$ for any $k \in \mathbb{R}_{\geq 0}$, σ_1^* is optimal for party 1. For a policy commitment $\mathbf{q} \in X$ of party 2 such that $k_2(\mathbf{q}) \leq \tilde{k}_2$, the largest policy on the $\mathbf{b}_1$-ray that generates a victory of party 1 is $\tilde{p}_1(\mathbf{q})\mathbf{b}_1$, and the same policy maximizes the dynamic utility of party 1. For a policy commitment $\mathbf{q} \in X$ of party 2 such that $k_2(\mathbf{q}) > \tilde{k}_2$, running with policy $\mathbf{b}_1$ guarantees the electoral victory for party 1, and the same policy maximizes the dynamic utility of party 1.

From the proof of Proposition 1 we know that $u_2(k\mathbf{b}_2)+\delta V_2^2(k\mathbf{b}_2|\sigma')$ is increasing in k for $k \in [0,\hat{k}_2]$ and decreasing in k for $k \geq \hat{k}_2$. Since $V_2^2(k\mathbf{b}_2|\sigma^*) = V_2^2(k\mathbf{b}_2|\sigma')$ for $k \in [0,\hat{k}_2]$ and $V_2^2(k\mathbf{b}_2|\sigma^*) \leq V_2^2(k\mathbf{b}_2|\sigma')$ for $k > \hat{k}_2$, σ_2^* is optimal for party 2. For a policy commitment $\mathbf{q} \in X$ of party 1 such that $k_2(\mathbf{q}) \leq \hat{k}_2$, the largest policy on the $\mathbf{b}_2$-ray that generates the victory of party 2 is $k_2(\mathbf{q})\mathbf{b}_2$, and the same policy maximizes the dynamic utility of party 2. For a policy commitment $\mathbf{q} \in X$ of party 1 such that $k_2(\mathbf{q}) > \hat{k}_2$, running with policy $\hat{k}_2\mathbf{b}_2$ guarantees the electoral victory for party 2, and the same policy maximizes the dynamic utility of party 2.

It remains is to be shown that σ_v^* satisfies the definition of SMPE. From the definition, for any incumbent $j \in \{1,2\}$ committed to $\mathbf{q} \in X$ and the opposition $i = \{1,2\} \setminus \{j\}$ contesting elections with $\mathbf{p} \in X$, $\sigma_v^*(j,\mathbf{q},\mathbf{p}) = Yes$ if and only if

$$u_v(\mathbf{p}) + \delta_v V_v^i(\mathbf{p}|\sigma^*) \geq u_v(\mathbf{q}) + \delta_v V_v^j(\mathbf{q}|\sigma^*). \tag{A24}$$

Using σ_1^*, σ_2^* and $u_v(\mathbf{q}) = -d^2(\mathbf{q})$ for any $\mathbf{q} \in X$, straightforward algebra gives

$$\begin{aligned} V_v^1(\mathbf{q}|\sigma^*) &= \begin{cases} -\frac{d^2(\mathbf{q})}{1-\delta_v} & \text{for } k_2(\mathbf{q}) \leq \hat{k}_2 \\ -\frac{(\hat{k}_2 b_2)^2}{1-\delta_v} & \text{for } k_2(\mathbf{q}) > \hat{k}_2 \end{cases} \\ V_v^2(\mathbf{q}|\sigma^*) &= \begin{cases} -\frac{d^2(\mathbf{q})}{1-\delta_v} & \text{for } k_2(\mathbf{q}) \leq \tilde{k}_2 \\ -b_1^2 - \delta_v\frac{(\hat{k}_2 b_2)^2}{1-\delta_v} & \text{for } k_2(\mathbf{q}) > \tilde{k}_2 \end{cases} \end{aligned} \tag{A25}$$

which can be used to derive σ_v^* as stated in (A22) and (A23). This concludes the proof that σ^* constitutes an SMPE.

Part 1 of the proposition, that elections are always contested and incumbents always defeated under σ^*, has already been noted. Part 2 claims that policies converge to identical alternation under σ' and σ^*. From Proposition 1, this means alternation between $k_1(\mathbf{q})\mathbf{b}_1$ and $k_2(\mathbf{q})\mathbf{b}_2$ when $k_2(\mathbf{q}) \leq \hat{k}_2$ and alternation between $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$ and $\hat{k}_2\mathbf{b}_2$ when $k_2(\mathbf{q}) > \hat{k}_2$. When $k_2(\mathbf{q}) \leq \hat{k}_2$, σ^* clearly generates the identical alternation since $\sigma_2^* = \sigma_2'$ and $\sigma_1^*(\mathbf{q}) = k_1(\mathbf{q})\mathbf{b}_1$. When $k_2(\mathbf{q}) > \hat{k}_2$, the first time party 2 contests elections it does so with $\sigma_2^*(\mathbf{q}) = \hat{k}_2\mathbf{b}_2$ and from then on the equilibrium policies alternate between $k_1(\hat{k}_2\mathbf{b}_2)\mathbf{b}_1$ and $\hat{k}_2\mathbf{b}_2$. To show part 3, taking derivatives with respect to α and b_i for $i \in \{1, 2\}$ of the expressions in (A25) shows that $\frac{\partial}{\partial\alpha}V_v^j(\mathbf{q}|\sigma^*) \geq 0$ and $\frac{\partial}{\partial b_i}V_v^j(\mathbf{q}|\sigma^*) \leq 0$ for any $i \in \{1, 2\}$, any $j \in \{1, 2\}$ and any $\mathbf{q} \in X$. At the same time $\frac{\partial}{\partial b_1}V_v^2(\mathbf{q}|\sigma^*) < 0$ when $k_2(\mathbf{q}) > \tilde{k}_2$, $\frac{\partial}{\partial b_2}V_v^1(\mathbf{q}|\sigma^*) < 0$ when $k_2(\mathbf{q}) > \hat{k}_2$ and $\frac{\partial}{\partial\alpha}V_v^1(\mathbf{q}|\sigma^*) > 0$ when $k_2(\mathbf{q}) > \hat{k}_2$ and $\delta > 0$. □

A2.3 Unique SPE with Finite Horizon

In this section, we analyze a finite horizon version of the model studied so far. We show that there exists a unique subgame perfect equilibrium (SPE) and describe the policy dynamics generated by this equilibrium strategy profile.

Assume the parties compete in T consecutive elections, where $T < \infty$. For simplicity, we assume that $b_1 = b_2$ and that the two parties have lexicographic preferences over office rents and policy outcomes. This implies that the opposition party, for any incumbent's policy commitment $\mathbf{q} \in X$, contests the election with a winning policy, if such a policy exists. Denote $\kappa = \frac{1+\delta\cos\alpha}{1+\delta}$ and note that $\kappa \in (0, 1]$ for any $\delta \in [0, 1)$ and $\alpha \in [0, \pi]$. We use the notation $\mathbf{p}_i(\mathbf{q}|\hat{k}_i)$ for the simple strategies from Definition 3, in order to make explicit their dependence on $\hat{k}_i$.

In any SPE, the voting strategy of the representative voter in any period has to satisfy, for any incumbent party $j \in \{1, 2\}$, any incumbent's policy commitment $\mathbf{q} \in X$ and any electoral platform of the opposition party $\mathbf{p} \in X$, $\sigma_v(j, \mathbf{q}, \mathbf{p}) = Yes$ if and only if $d(\mathbf{p}) \leq d(\mathbf{q})$.

We proceed by backward induction. Consider the last period T. For any incumbent's policy commitment $\mathbf{q} \in X$, the opposition party i contests elections with policy $\mathbf{p}^* \in \arg\max_{\mathbf{p}\in\{\mathbf{x}\in X|d(\mathbf{x})\leq d(\mathbf{q})\}} -d^2(\mathbf{p}, \mathbf{b}_i)$. Clearly, $\mathbf{p}^*$ has to lie on the $\mathbf{b}_i$-ray and, hence, can be written as $k\mathbf{b}_i$ for some $k \geq 0$. Denote $\tilde{U}_{i,T}(k) = -d^2(k\mathbf{b}_i, \mathbf{b}_i)$. Then the optimization problem of party i can be written as $\max_{k\in[0,k_i(\mathbf{q})]} \tilde{U}_{i,T}(k)$. This problem has a unique solution $k = 1$ if $k_i(\mathbf{q}) \geq 1$ and $k = k_i(\mathbf{q})$ if $k_i(\mathbf{q}) \leq 1$. In other words, $\mathbf{p}_i(\mathbf{q}|1)$ is the uniquely optimal strategy of the opposition party i

for all $i \in \{1, 2\}$ and $\mathbf{q} \in X$. Denote $\mathbf{p}_{i,T}(\mathbf{q}) = \mathbf{p}_i(\mathbf{q}|1)$ for $i \in \{1, 2\}$.

Consider the period $T-1$. For any incumbent's policy commitment $\mathbf{q} \in X$, the opposition party i contests elections with a policy on the $\mathbf{b}_i$-ray because both parties use simple strategies in period T that depend on $d(\mathbf{q})$ but not on the exact location of $\mathbf{q}$. The optimization problem of the opposition party i is $\max_{k \in [0,k_i(\mathbf{q})]} \tilde{U}_{i,T-1}(k)$ where $\tilde{U}_{i,T-1}(k) = -d^2(k\mathbf{b}_i, \mathbf{b}_i) - d^2(\mathbf{p}_{-i,T}(k\mathbf{b}_i), \mathbf{b}_i)\delta$. Because $\mathbf{p}_{-i,T}(k\mathbf{b}_i) = k\mathbf{b}_{-i}$ for $k \in [0, 1]$ and $\mathbf{p}_{-i,T}(k\mathbf{b}_i) = \mathbf{b}_{-i}$ for $k \geq 1$,

$$\begin{aligned} \tilde{U}_{i,T-1}(k) &= -d^2(k\mathbf{b}_i, \mathbf{b}_i) - d^2(k\mathbf{b}_{-i}, \mathbf{b}_i)\delta \\ \tilde{U}_{i,T-1}(k) &= -d^2(k\mathbf{b}_i, \mathbf{b}_i) - d^2(\mathbf{b}_{-i}, \mathbf{b}_i)\delta \end{aligned} \tag{A26}$$

when $k \in [0, 1]$ and $k \geq 1$ respectively. An identical argument to the one used to prove Lemma A1 then shows that $\tilde{U}_{i,T-1}(k)$ has a unique maximum at $k = \kappa$, is increasing on $[0, \kappa]$ and decreasing on $[\kappa, \infty)$. Thus, the optimization problem of the opposition party i has a unique solution $k = \kappa$ if $k_i(\mathbf{q}) \geq \kappa$ and $k = k_i(\mathbf{q})$ if $k_i(\mathbf{q}) \leq \kappa$. In other words, $\mathbf{p}_i(\mathbf{q}|\kappa)$ is the uniquely optimal strategy of the opposition party i for all $i \in \{1, 2\}$ and $\mathbf{q} \in X$. Denote $\mathbf{p}_{i,T-1}(\mathbf{q}) = \mathbf{p}_i(\mathbf{q}|\kappa)$ for $i \in \{1, 2\}$.

Consider now the period $T-2$. Without repeating the obvious details, for any incumbent's policy commitment $\mathbf{q} \in X$, the optimization problem of the opposition party i writes $\max_{k \in [0,k_i(\mathbf{q})]} \tilde{U}_{i,T-2}(k)$ where

$$\tilde{U}_{i,T-2}(k) = -d^2(k\mathbf{b}_i, \mathbf{b}_i) - d^2(\mathbf{p}_{-i,T-1}(k\mathbf{b}_i), \mathbf{b}_i)\delta - d^2(\mathbf{p}_{i,T}(\mathbf{p}_{-i,T-1}(k\mathbf{b}_i)), \mathbf{b}_i)\delta^2. \tag{A27}$$

Because $\mathbf{p}_{-i,T-1}(k\mathbf{b}_i) = k\mathbf{b}_{-i}$ for $k \in [0, \kappa]$ and $\mathbf{p}_{-i,T-1}(k\mathbf{b}_i) = \kappa\mathbf{b}_{-i}$ for $k \geq \kappa$, we have $\mathbf{p}_{i,T}(\mathbf{p}_{-i,T-1}(k\mathbf{b}_i)) = k\mathbf{b}_i$ if $k \in [0, \kappa]$ and $\mathbf{p}_{i,T}(\mathbf{p}_{-i,T-1}(k\mathbf{b}_i)) = \kappa\mathbf{b}_i$ if $k \geq \kappa$. Hence

$$\begin{aligned} \tilde{U}_{i,T-2}(k) &= -d^2(k\mathbf{b}_i, \mathbf{b}_i) - d^2(k\mathbf{b}_{-i}, \mathbf{b}_i)\delta - d^2(k\mathbf{b}_i, \mathbf{b}_i)\delta^2 \\ \tilde{U}_{i,T-2}(k) &= -d^2(k\mathbf{b}_i, \mathbf{b}_i) - d^2(\kappa\mathbf{b}_{-i}, \mathbf{b}_i)\delta - d^2(\kappa\mathbf{b}_i, \mathbf{b}_i)\delta^2 \end{aligned} \tag{A28}$$

when $k \in [0, \kappa]$ and $k \geq \kappa$ respectively. We first note that $\tilde{U}_{i,T-2}(k)$ is increasing on $[0, \kappa]$, which follows from $\tilde{U}'_{i,T-2}(k) = \tilde{U}'_{i,T-1}(k) - 2b_i^2\delta^2(k-1) > \tilde{U}'_{i,T-1}(k) > 0$ for any $k \in (0, \kappa)$. Furthermore, $\tilde{U}_{i,T-2}(k)$ is clearly increasing on $[\kappa, 1]$ and decreasing on $[1, \infty)$. Thus the optimization problem of the opposition party i has a unique solution $k = 1$ if $k_i(\mathbf{q}) \geq 1$ and $k = k_i(\mathbf{q})$ if $k_i(\mathbf{q}) \leq 1$. In other words, $\mathbf{p}_i(\mathbf{q}|1)$ is the uniquely optimal strategy of the opposition party i for all $i \in \{1, 2\}$ and $\mathbf{q} \in X$. Denote $\mathbf{p}_{i,T-2}(\mathbf{q}) = \mathbf{p}_i(\mathbf{q}|1)$ for $i \in \{1, 2\}$.

Consider the period $T-3$. For any $\mathbf{q} \in X$, the optimization problem of the opposition party i writes $\max_{k \in [0,k_i(\mathbf{q})]} \tilde{U}_{i,T-3}(k)$, where similar arguments as in the previous period show

$$
\begin{aligned}
\tilde{U}_{i,T-3}(k) &= -d^2(k\mathbf{b}_i,\mathbf{b}_i) - d^2(k\mathbf{b}_{-i},\mathbf{b}_i)\delta - d^2(k\mathbf{b}_i,\mathbf{b}_i)\delta^2 - d^2(k\mathbf{b}_{-i},\mathbf{b}_i)\delta^3 \\
\tilde{U}_{i,T-3}(k) &= -d^2(k\mathbf{b}_i,\mathbf{b}_i) - d^2(k\mathbf{b}_{-i},\mathbf{b}_i)\delta - d^2(\kappa\mathbf{b}_i,\mathbf{b}_i)\delta^2 - d^2(\kappa\mathbf{b}_{-i},\mathbf{b}_i)\delta^3
\end{aligned}
\tag{A29}
$$

when $k \in [0,\kappa]$ and $k \geq \kappa$ respectively. Denoting by $\upsilon_i(k) = -d^2(k\mathbf{b}_i,\mathbf{b}_i) - d^2(k\mathbf{b}_{-i},\mathbf{b}_i)\delta$, $\tilde{U}_{i,T-3}(k) = (1+\delta^2)\upsilon_i(k)$ if $k \in [0,\kappa]$ and $\tilde{U}_{i,T-3}(k) = \upsilon_i(k) + c$ if $k \geq \kappa$, where c is constant in k. Because $\upsilon_i(k)$ is increasing in k on $[0,\kappa]$ and decreasing in k on $[\kappa,\infty)$, $\tilde{U}_{i,T-3}(k)$ has a unique maximum at $k=\kappa$. By the now familiar arguments, $\mathbf{p}_i(\mathbf{q}|\kappa)$ is the uniquely optimal strategy of the opposition party i for all $i \in \{1,2\}$ and $\mathbf{q} \in X$. Denote $\mathbf{p}_{i,T-3}(\mathbf{q}) = \mathbf{p}_i(\mathbf{q}|\kappa)$ for $i \in \{1,2\}$.

So far, we have shown that it is uniquely optimal for the opposition party i to contest elections with $\mathbf{p}_i(\mathbf{q}|\hat{k}_i)$, where $\hat{k}_i = 1$ for T and $T-2$ and $\hat{k}_i = \kappa$ for $T-1$ and $T-3$. Suppose that this patterns repeats for periods up to $T-s+2$ and $T-s+1$ where s is even.

Consider period $T-s$. We need to show that it is uniquely optimal for the opposition party i to contest elections with $\mathbf{p}_i(\mathbf{q}|1)$ for all $i \in \{1,2\}$ and $\mathbf{q} \in X$. Because in $T-s+1$ the opposition party contests elections with $\mathbf{p}_i(\mathbf{q}|\kappa)$,

$$
\begin{aligned}
\tilde{U}_{i,T-s}(k) &= \upsilon_i(k)\sum_{t=0}^{\frac{s-2}{2}}\delta^{2t} - d^2(k\mathbf{b}_i,\mathbf{b}_i)\delta^s \\
\tilde{U}_{i,T-s}(k) &= -d^2(k\mathbf{b}_i,\mathbf{b}_i) + c
\end{aligned}
\tag{A30}
$$

when $k \in [0,\kappa]$ and $k \geq \kappa$ respectively. Because $\upsilon_i(k)$ is increasing in k on $[0,\kappa]$ and decreasing in k on $[\kappa,\infty)$, and because $-d^2(k\mathbf{b}_i,\mathbf{b}_i)$ is increasing in k on $[0,1]$ and decreasing in k on $[1,\infty)$, $\tilde{U}_{T-s}(k)$ has a unique maximum $k=1$. Hence, it is uniquely optimal for the opposition party i to contest elections with $\mathbf{p}_i(\mathbf{q}|1)$ for all $i \in \{1,2\}$ and $\mathbf{q} \in X$.

Consider $T-s-1$. We need to show that it is uniquely optimal for the opposition party i to contest elections with $\mathbf{p}_i(\mathbf{q}|\kappa)$ for all $i \in \{1,2\}$ and $\mathbf{q} \in X$. Because in $T-s$ the opposition party contests elections with $\mathbf{p}_i(\mathbf{q}|1)$ and in $T-s+1$ with $\mathbf{p}_i(\mathbf{q}|\kappa)$,

$$
\begin{aligned}
\tilde{U}_{i,T-s-1}(k) &= \upsilon_i(k)\sum_{t=0}^{\frac{s}{2}}\delta^{2t} \\
\tilde{U}_{i,T-s-1}(k) &= \upsilon_i(k) + c
\end{aligned}
\tag{A31}
$$

when $k \in [0,\kappa]$ and $k \geq \kappa$ respectively. Because $\upsilon_i(k)$ is increasing in k on $[0,\kappa]$ and decreasing in k on $[\kappa,\infty)$, $\tilde{U}_{T-s-1}(k)$ has a unique maximum $k=\kappa$. Hence, it is uniquely optimal for the opposition party i to contest elections with $\mathbf{p}_i(\mathbf{q}|\kappa)$ for all $i \in \{1,2\}$ and $\mathbf{q} \in X$.

To summarize, there exists a unique SPE in which the opposition party i contests elections with $\mathbf{p}_i(\mathbf{q}|\hat{k}_i)$ for all $i \in \{1,2\}$ and $\mathbf{q} \in X$. $\hat{k}_i = 1$ for all periods $T-s$ with $s \in \mathbb{N}_{\geq 0}$ and s even and $\hat{k}_i = \kappa$ for all periods $T-s$ with $s \in \mathbb{N}_{\geq 1}$ and s odd. With an appropriately chosen initial-period incumbent party, the SPE just described generates identical policy dynamics to the one generated by the SSMPE from Proposition 1, for any incumbent's policy commitment $\mathbf{q} \in X$. When party 2 is at the opposition in the initial period of the infinite horizon model, both parties moderate, if at the opposition, in the initial period of the finite horizon model with T even. When party 1 is at the opposition in the initial period of the infinite horizon model, none of the parties moderates, if at the opposition, in the initial period of the finite horizon model with T odd.