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Dynamic Free Riding with Irreversible Investments: On-line Appendix

Abstract

In this appendix we present the proofs omitted in “Dynamic Free Riding with Irreversible Investments” by Marco Battaglini, Salvatore Nunnari and Thomas Palfrey.

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1 Proof of Proposition 1

Proposition 1. *For any d, δ, n and $y^o \in \left[[u']^{-1}(1 - \delta(1 - d)), [u']^{-1}(1 - \delta(1 - \frac{d}{n})) \right]$, there is an equilibrium with steady state y^o in an irreversible economy. In all these equilibria convergence is monotonic and gradual.*

Define $y^*(\delta, d, n) = [u']^{-1}(1 - \delta(1 - d)/n)$ and $y^{**}(\delta, d, n) = [u']^{-1}(1 - \delta(1 - d/n))$: these are the points at which

$$y'(g) = \frac{1 - d - \frac{n(1 - u'(g))}{\delta}}{1 - n} \quad (1)$$

is, respectively, zero and one. Define $\bar{y}(d, \delta) = [u']^{-1}(1 - \delta(1 - d))$: this is the point at which (1) is equal to $1 - d$. Note that $y^*(\delta, d, n) < \bar{y}(d, \delta)$ and $\bar{y}(d, \delta) < y^{**}(\delta, d, n)$. Moreover, since we are assuming that the planner interior solution is feasible ($y_P^*(\delta, d, n) < W/d$), we have $y^{**}(\delta, d, n) < W/d$. To construct an equilibrium with steady state $y^o \in [\bar{y}(d, \delta), y^{**}(\delta, d, n)]$ we proceed in 3 steps.

Step 1. We first construct the strategies associated to a generic y^o . For a generic $y^o \in [\bar{y}(d, \delta), y^{**}(\delta, d, n)]$, let $\tilde{y}(g | y^o)$ be the solution of the differential equation (1) when we require the initial condition: $\tilde{y}(y^o | y^o) = y^o$. Given y^o , moreover, let us define the two thresholds $g^3(y^o) = y^o/(1 - d)$ and $g^2(y^o) = \max \{ \min_{g \geq 0} \{ g | \tilde{y}(g | y^o) \leq W + (1 - d)g \}, y^*(\delta, d, n) \}$. In words, the second threshold is the largest point between the point at which $\tilde{y}(g | y^o)$ crosses from below $W + (1 - d)g$, and $y^*(\delta, d, n)$ (see Figure 1 in the paper for an example). It is easy to verify that, by construction, $g^3(y^o) \geq \bar{y}(d, \delta)$; moreover, $\tilde{y}(g | y^o) \in ((1 - d)g, W + (1 - d)g)$ with $\tilde{y}'(g | y^o) \in [0, 1]$ and $\tilde{y}''(g | y^o) \geq 0$ in $[g^2(y^o), y^o]$. For any $y^o \in [\bar{y}(d, \delta), y^{**}(\delta, d, n)]$, we now define the investment function as follows:

$$y(g | y^o) = \begin{cases} \min \{ W + (1 - d)g, \tilde{y}(g^2(y^o) | y^o) \} & g \leq g^2(y^o) \\ \tilde{y}(g | y^o) & g^2(y^o) < g \leq y^o \\ y^o & y^o < g \leq g^3(y^o) \\ (1 - d)g & g > g^3(y^o) \end{cases} \quad (2)$$

Note that when depreciation is zero, then $g^3(y^o) = y^o$ and $y'(g | y^o) = 1$ at $g = y^o$: so (2) coincides exactly with the investment function illustrated in Figure 1 in the paper. For future reference, define $g^1(y^o) = \max \{ 0, (\tilde{y}(g^2(y^o) | y^o) - W) / (1 - d) \}$. This is the point at which $W + (1 - d)g = \tilde{y}(g^2(y^o) | y^o)$, if positive. Since $\tilde{y}(g^2(y^o) | y^o) < W + (1 - d)g^2(y^o)$, $g^1(y^o) \in [0, g^2(y^o)]$. We have:

Lemma A.1. $y(g | y^o) \in [g^2(y^o), y^o]$ for $g \in [g^2(y^o), y^o]$.

Proof. Because $y(g|y^\circ)$ is monotonic non-decreasing in $g \in [g^2(y^\circ), y^\circ]$, for any $g \in [g^2(y^\circ), y^\circ]$ we have $y(g|y^\circ) \in [y(g^2(y^\circ)|y^\circ), y^\circ]$. Since $y(g|y^\circ)$ has slope lower than one in $[g^2(y^\circ), y^\circ]$ and $y(y^\circ|y^\circ) = y^\circ$ for $y^\circ \geq g^2(y^\circ)$, we must have $y(g^2(y^\circ)|y^\circ) \geq g^2(y^\circ)$, so $y(g|y^\circ) \geq g^2(y^\circ)$ for $g \in [g^2(y^\circ), y^\circ]$. Similarly, $y(y^\circ|y^\circ) = y^\circ$ implies $y(g|y^\circ) \leq y^\circ$ for $g \in [g^2(y^\circ), y^\circ]$. ■

Step 2. We now construct the value functions corresponding to each steady state y° . For $g \in [g^2(y^\circ), y^\circ]$ define the value function recursively as

$$v(g|y^\circ) = \frac{W + (1-d)g - y(g|y^\circ)}{n} + u(y(g|y^\circ)) + \delta v(y(g|y^\circ)). \quad (3)$$

By Theorem 3.3 in Stokey, Lucas, and Prescott (1989), the right hand side of (3) is a contraction: it defines a unique, continuous and differentiable value function $v(g|y^\circ)$ for this interval of g . (Differentiability follows from the differentiability of $y(g|y^\circ)$). Note that $y(g|y^\circ) = \tilde{y}(g|y^\circ)$ for any $g \in [g^2(y^\circ), y^\circ]$ and, by Lemma A.1, $\tilde{y}(g|y^\circ) \in [g^2(y^\circ), y^\circ]$ for $g \in [g^2(y^\circ), y^\circ]$. From the definition of $\tilde{y}(g|y^\circ)$ and the discussion in Section 4 in the paper, it follows that $u'(g) + \delta v'(g; y^\circ) = 1$ for any $g \in [g^2(y^\circ), y^\circ]$. In the rest of the state space we define the value function recursively. In $[g^1(y^\circ), g^2(y^\circ)]$, if $g^1(y^\circ) < g^2(y^\circ)$, the value function is defined as:

$$v(g|y^\circ) = \frac{W + (1-d)g - y(g^2(y^\circ)|y^\circ)}{n} + u(y(g^2(y^\circ)|y^\circ)) + \delta v(y(g^2(y^\circ)|y^\circ)) \quad (4)$$

where $v(y(g^2(y^\circ)|y^\circ))$ is well defined since $y(g^2(y^\circ)|y^\circ) \in [g^2(y^\circ), y^\circ]$.

Lemma A.2. For $g \in [g^1(y^\circ), y^\circ]$, $u(g) + \delta v(g|y^\circ)$ is concave with slope larger or equal than 1.

Proof. If $g^1(y^\circ) = g^2(y^\circ)$, the result is immediate. Assume therefore, $g^1(y^\circ) < g^2(y^\circ)$. In this case $g^2(y^\circ) = y^*(\delta, d, n)$. For any $g \in [g^1(y^\circ), g^2(y^\circ)]$, $y(g; y^\circ) = y(y^*(\delta, d, n)|y^\circ)$. So we have $v'(g|y^\circ) = (1-d)/n$ implying: $u'(g) + \delta v'(g|y^\circ) = u'(g) + \delta(1-d)/n > 1$ since $g \leq g^2(y^\circ) = y^*(\delta, d, n)$. ■

Consider $g < g^1(y^\circ)$. In $[g_{-1}, g^1(y^\circ)]$ the value function is defined as:

$$v(g|y^\circ) = u(W + (1-d)g) + \delta v(W + (1-d)g|y^\circ) \quad (5)$$

where $g_{-1} = \max\{0, [g^1(y^\circ) - W] / (1-d)\}$. Assume that we have defined the value function in $g \in [g_{-t}, g_{-(t-1)}]$ as v_{-t} , for all t such that $g_{-(t-1)} > 0$. Then we can define $v_{-(t+1)}$ as (5) in $[g_{-(t+1)}, g_{-t}]$ with $g_{-(t+1)} = [g_{-t} - W] / (1-d)$.

Lemma A.3. For $g \in [0, y^\circ]$, $u(g) + \delta v(g|y^\circ)$ is concave with slope greater than or equal than 1.

Proof. We prove this by induction on t . Consider now the interval $[[g^1(y^\circ) - W] / (1-d), g^1(y^\circ)]$. In this range we have $v'(g|y^\circ) = [u'(W + (1-d)g) + \delta v'(W + (1-d)g|y^\circ)](1-d) \geq 1-d$,

since $W + (1 - d)g \in [g^1(y^o), y^o]$. It follows that for $g \in [[g^1(y^o) - W] / (1 - d), g^1(y^o)]$: $u'(g) + \delta v'(g|y^o) \geq u'(g) + \delta(1 - d) > 1$. Where the last inequality follows from the fact that $g \leq g^1(y^o) < \bar{y}(\delta, d)$. We conclude that $u'(g) + \delta v'_{-1}(g|y^o)$ is concave, it has derivative larger than 1. Assume that we have shown that for $g \in [g_{-t}, g^3(y^o)]$, $u(g) + \delta v_{-t}(g|y^o)$ is concave and $u'(g) + \delta v'_{-t}(g|y^o) > 1$. Consider in $g \in [g_{-(t+1)}, g_{-t}]$. We have:

$$v'(g|y^o) = [u'(W + (1 - d)g) + \delta v'(W + (1 - d)g|y^o)](1 - d) \geq 1 - d$$

since $W + (1 - d)g \geq [g_{-t}, g^3(y^o)]$. So $u'(g) + \delta v'(g|y^o) \geq u'(g) + \delta(1 - d) \geq 1$. By the same argument as above, moreover, v is concave at g_{-t} . We conclude that for any $g \leq g^1$, $u(g) + \delta v(g|y^o)$ is concave and it has slope larger than 1. ■

For $g \in (y^o, g^3(y^o)]$ we define the value function as: $v(g|y^o) = \frac{W + (1-d)g - y^o}{n} + u(y^o) + \delta v(y^o|y^o)$.

Lemma A.4. For $g \leq g^3(y^o)$, $u(g) + \delta v(g|y^o)$ is concave with slope less than or equal than 1.

Proof. For $g \in (y^o, g^3(y^o)]$, $v'(g|y^o) = (1 - d)/n$. Since $g \geq y^o \geq y^*(\delta, d, n)$, we have $u'(g) + \delta v'(g|y^o) = u'(g) + \delta(1 - d)/n < 1$. Previous lemmas imply $u(g) + \delta v(g|y^o)$ is concave and has slope greater than or equal than 1 for $g \leq g^3(y^o)$. ■

Finally consider $g > g^3(y^o)$.

Lemma A.5. For any $g \geq g^3(y^o)$, $u(g) + \delta v(g|y^o)$ has slope less than or equal than 1.

Proof. In $g > g^3(y^o)$, we must have $(1 - d)g \in [y^o, g^3(y^o)]$. From the proof of Lemma A.5 we know that $u'(g) + \delta v'(g) < 1$ for $g \in [y^o, g^3(y^o)]$, so we have:

$$v'(g) = (1 - d)[u'((1 - d)g) + \delta v'((1 - d)g)] < 1 - d$$

for $g > g^3(y^o)$. This fact implies that $u'(g) + \delta v'(g) < u'(g) + \delta(1 - d)$ for any $g > g^3(y^o)$. Since $g^3(y^o) > \bar{y}(\delta, d)$ we have $u'(g) + \delta(1 - d) < u'(\bar{y}(\delta, d)) + \delta(1 - d) = 1$ for $g > g^3(y^o)$. It follows that $v^*(g)$ is has slope lower than 1 in $g > g^3(y^o)$. ■

From Lemmata A1-A5 we conclude that $u(g) + \delta v(g|y^o)$ has a global maximum at any $g \in [g^3(y^o), y^o]$.

Step 3. Define $x(g|y^o) = [W + (1 - d)g - y(g|y^o)]/n$ and $i(g|y^o) = [y(g|y^o) - (1 - d)g]/n$ as the levels of per capita private consumption and investment, respectively. Note that by construction, $x(g|y^o) \in [0, W/n]$. We now establish that $y(g|y^o)$, $x(g|y^o)$ and the associated value function $v(g|y^o)$ defined in the previous steps constitute an equilibrium. The fact that $v(g|y^o)$ describes the expected continuation value to an agent follows by construction. To see that $y(g|y^o)$ is an optimal reaction function given $v(g|y^o)$, note that an agent solves the following

problem:

$$\max_y \left\{ \begin{array}{l} u(y) - y + \delta v(y) \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y(g), \quad y \geq \frac{(1-d)g}{n} + \frac{n-1}{n}y(g) \end{array} \right\} \quad (6)$$

where $y(g) = y(g|y^o)$. The investment function $y(g|y^o)$ satisfies the constraints of this problem if $y(g|y^o) \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y(g|y^o)$, so if $y(g|y^o) \leq W + (1-d)g$; and if $y(g|y^o) \geq \frac{(1-d)g}{n} + \frac{n-1}{n}y(g|y^o)$, so if $y(g|y^o) \geq (1-d)g$. Both conditions are automatically satisfied by construction. If $g < g^1(y^o)$, we have $u'(y) + \delta v'(y) \geq 1$ for all $y \in [(1-d)g, W + (1-d)g]$, so $y(g|y^o) = W + (1-d)g$ is optimal. If $g \geq g^3(y^o)$, $u'(y) + \delta v'(y) < 1$ for all $y \in [(1-d)g, W + (1-d)g]$, so $y(g|y^o) = (1-d)g$. In $g \in (g^1(y^o), g^3(y^o))$ a point maximizing $u(y) + \delta v(y)$ is feasible and chosen, so again $y(g|y^o)$ is an optimal choice. ■

2 Proof of Proposition 2

Propositon 2. *For any δ and n , we have that $|\bar{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n)| \rightarrow 0$ as $d \rightarrow 0$. Moreover, there is $\bar{d} > 0$ a such that for $d < \bar{d}$, all equilibrium paths are gradual.*

Consider a sequence $d^m \rightarrow 0$. For each d^m there is at least an associated equilibrium $y_m(g)$, $v_m(g)$ with steady state y_m^o . To prove the result we proceed in two steps. In Section 2.1 we prove that for any $\xi > 0$, there is a \tilde{m} such that for $m > \tilde{m}$, $\underline{y}_{IR}(\delta, d^m, n) \geq [u']^{-1}(1 - \delta) - \xi$. In Section 2.2 we prove that the steady state of any equilibrium can not be larger than $[u']^{-1}(1 - \delta(1 - d/n))$. Since, as shown in Proposition 1, $[u']^{-1}(1 - \delta(1 - d/n))$ is an equilibrium steady state for any $d \geq 0$ and it converges to $[u']^{-1}(1 - \delta)$, we must have $|\bar{y}_{IR}(\delta, d, n) - \underline{y}_{IR}(\delta, d, n)| \rightarrow 0$ as $d \rightarrow 0$. In Lemmata A.6 and A.7 presented in Section 2.2 we show that $y'(g) \in (0, 1)$ in a left neighborhood of the steady state y^o if $y^o > [u']^{-1}(1 - \delta(1 - d)/n)$. Since all equilibrium steady states converge to $[u']^{-1}(1 - \delta) > [u']^{-1}(1 - \delta/n)$, this implies that that convergence of g to the steady state is gradual in all equilibria if d is sufficiently small.

2.1 The lower bound

We prove the result by contradiction. Suppose to the contrary there is a sequence of steady states y_m^o , with associated equilibrium investment and value functions $y_m(g)$, $v_m(g)$, and an $\xi > 0$ such that $y_m^o < \bar{y}(0) - \xi$ for any arbitrarily large m , where $\bar{y}(d) = [u']^{-1}(1 - \delta(1 - d))$. Define $y_m^0(g) = y_m(g)$, and $y_m^j(g) = y_m(y_m^{j-1}(g))$ and consider a marginal deviation from the steady state

from y_m^0 to $y_m^0 + \Delta$. By the irreversibility constraint we have $y_m(g) \geq (1 - d^m)g$. Using this property and the fact that y_m^0 is a steady state, so $y_m^j(y_m^0) = y_m^0$, we have:

$$y_m(y_m^0 + \Delta) - y_m(y_m^0) \geq (1 - d^m)(y_m^0 + \Delta) - y_m^0 = (1 - d^m)\Delta - d^m y_m^0$$

This implies that, as $m \rightarrow \infty$, for any given Δ : $[y_m(y_m^0 + \Delta) - y_m^0] / \Delta \geq 1 + o_1(d^m)$ where $o_1(d^m) \rightarrow 0$ as $m \rightarrow 0$. We now show with an inductive argument that a similar property holds for all iterations $y_m^j(y_m^0)$. Assume we have shown that: $[y_m^{j-1}(y_m^0 + \Delta) - y_m^0] / \Delta \geq 1 + o_{j-1}(d^m)$ where $o_{j-1}(d^m) \rightarrow 0$ as $m \rightarrow 0$. We must have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^j(y_m^0) \geq (1 - d^m)y_m^{j-1}(y_m^0 + \Delta) - y_m^0$. We therefore have: $y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0 \geq y_m^{j-1}(y_m^0 + \Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0 + \Delta)$ so we have:

$$\frac{y_m(y_m^{j-1}(y_m^0 + \Delta)) - y_m^0}{\Delta} \geq \frac{y_m^{j-1}(y_m^0 + \Delta) - y_m^0}{\Delta} - \frac{d^m y_m^{j-1}(y_m^0 + \Delta)}{\Delta} \geq 1 + o_j(d^m) \quad (7)$$

where $o_j(d^m) = o_{j-1}(d^m) - \frac{d^m y_m^{j-1}(y_m^0 + \Delta)}{\Delta}$, so $o_j(d^m) \rightarrow 0$ as $m \rightarrow 0$.

We can write the value function after the deviation to $y_m^0 + \Delta$ as:

$$V(y_m^0 + \Delta) = \sum_{j=0}^{\infty} \delta^{j-1} \left[\frac{W + (1 - d^m)y_m^{j-1}(y_m^0 + \Delta) - y_m^j(y_m^0 + \Delta)}{n} + u(y_m^j(y_m^0 + \Delta)) \right]$$

For any given function $f(x)$, define $\Delta f(x) = f(x + \Delta) - f(x)$. We can write:

$$\begin{aligned} \Delta V(y_m^0) / \Delta &= \sum_{j=0}^{\infty} \delta^{j-1} \left[\frac{(1-d^m)\Delta y_m^{j-1}(y_m^0) / \Delta - \Delta y_m^j(y_m^0) / \Delta}{n} \right. \\ &\quad \left. + [u'(y_m^0) + o(\Delta)] \Delta y_m^j(y_m^0) / \Delta \right] \\ &\geq \sum_{j=0}^{\infty} \delta^{j-1} \left[\frac{(1-d^m)(1+o_{j-1}(d^m)) - (1+o_j(d^m))}{n} \right. \\ &\quad \left. + [u'(y_m^0) + o(\Delta)] (1 + o_j(d^m)) \right] \end{aligned} \quad (8)$$

where $o(\Delta) \rightarrow 0$ as $\Delta \rightarrow 0$. In the first equality we use the fact that if we choose Δ small, since $y_m(g)$ is continuous, $\Delta y_m^j(y_m^0)$ is small as well. This implies that

$$(u(y_m^j(y_m^0 + \Delta)) - u(y_m^j(y_m^0))) / [y_m^j(y_m^0 + \Delta) - y_m^j(y_m^0)]$$

converges to $u'(y_m^j(y_m^0))$ as $\Delta \rightarrow 0$. The inequality in 8 follows from (7). Given Δ , as $m \rightarrow \infty$, we therefore have $\lim_{m \rightarrow \infty} \Delta V(y_m^0) / \Delta \geq \frac{u'(y_m^0) + o(\Delta)}{1 - \delta}$. We conclude that for any $\varepsilon > 0$, there must be a Δ_ε such that for any $\Delta \in (0, \Delta_\varepsilon)$ there is a m_Δ guaranteeing that $\Delta V(y_m^0) / \Delta \geq \frac{u'(y_m^0)}{1 - \delta} - \varepsilon$ for $m > m_\Delta$. After a marginal deviation to $y_m^0 + \Delta$, therefore, the change in agent's objective function is:

$$u'(y_m^0) + \delta \Delta V(y_m^0) / \Delta - 1 \geq \frac{u'(y_m^0)}{1 - \delta} - \delta \varepsilon - 1$$

for m sufficiently large. A necessary condition for the un-profitability of a deviation from y_m^0 to $y_m^0 + \Delta$ is therefore: $y_m^0 \geq [u']^{-1}(1 - \delta + \delta\varepsilon(1 - \delta))$. Since ε can be taken to be arbitrarily small, for an arbitrarily large m , this condition implies $y_m^0 \geq \bar{y}(0) - \xi/2$, which contradicts $y_m^0 < \bar{y}(0) - \xi$. We conclude that $\underline{y}_{IR}(\delta, d, n) \rightarrow \bar{y}(0)$ as $d \rightarrow 0$.

2.2 The upper bound

Suppose to the contrary that there is stable steady state at $y^\circ > [u']^{-1}(1 - \delta(1 - d/n))$. We must have $y^\circ \in \left([u']^{-1}(1 - \delta(1 - d/n)), W/d\right]$, since it is not feasible for a steady state to be larger than W/d . Consider a left neighborhood of y° , $N_\varepsilon(y^\circ) = (y^\circ - \varepsilon, y^\circ)$. The value function can be written in $g \in N_\varepsilon(y^\circ)$ as:

$$v(g) = u(y(g)) + \delta v(y(g)) - y(g) + \frac{W + (1-d)g}{n} + (1 - 1/n)y(g) \quad (9)$$

where $y(g)$ is the equilibrium strategy associated to y° . In $N_\varepsilon(y^\circ)$ the constraint $y \geq \frac{1-d}{n}g + \frac{n-1}{n}y(g)$ cannot be binding (else we would have $y(g) = (1-d)g$, but this is not possible in a neighborhood of $y^\circ > 0$). We consider two cases.

Case 1. Suppose first that $y^\circ < W/d$. We must therefore have that $y(g) < W + (1-d)g$ in $N_\varepsilon(y^\circ)$, so the constraint $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y$ is not binding. The solution is in the interior of the constraint set of (6), and the objective function $u(y(g)) + \delta v(y(g)) - y(g)$ is constant for $g \in N_\varepsilon(y^\circ)$.

Lemma A.6. *If $y^\circ > [u']^{-1}(1 - \delta(1 - d)/n)$, then there is a left neighborhood $N_\varepsilon(y^\circ)$ in which $y(g)$ is not constant.*

Proof. Suppose to the contrary that, for any $N_\varepsilon(y^\circ)$, there is an interval in $N_\varepsilon(y^\circ)$ in which $y(g)$ is constant. Using the expression for $v(g)$ presented above, we must have $v'(g) = (1-d)/n$ for any g in this interval. Since $N_\varepsilon(y^\circ)$ is arbitrary, then we must have a sequence $g^m \rightarrow y^\circ$ such that $v'(g^m) = (1-d)/n \forall m$. We can therefore write:

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \frac{v(y^\circ) - v(y^\circ - \Delta)}{\Delta} &= \lim_{\Delta \rightarrow 0} \lim_{m \rightarrow \infty} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} \\ &= \lim_{m \rightarrow \infty} \lim_{\Delta \rightarrow 0} \frac{v(g^m) - v(g^m - \Delta)}{\Delta} = \frac{1-d}{n} \end{aligned}$$

where the second equality follows from the continuity of $v(g)$. This implies that $v^-(y^\circ)$, left derivative of $v(g)$ at y° , is well defined and equal to $\frac{1-d}{n}$. Consider now a marginal reduction of g at y° . The change in utility is (as $\Delta \rightarrow 0$):

$$\begin{aligned} \Delta U(y^\circ) &= u(y^\circ - \Delta) - u(y^\circ) + \delta [v(y^\circ - \Delta) - v(y^\circ)] + \Delta \\ &= \left[1 - \left(u'(y^\circ) + \delta \frac{1-d}{n} \right) \right] \Delta \end{aligned}$$

In order to have $\Delta U(y^o) \leq 0$, we must have $u'(y^o) + \delta(1-d)/n \geq 1$. This implies $y^o \leq [u']^{-1}(1 - \delta(1-d)/n)$, a contradiction. Therefore, if there is stable steady state at $y^o > [u']^{-1}(1 - \delta(1-d)/n)$, then $y(g)$ is not constant in $N_\varepsilon(y^o)$. ■

Lemma A.6 implies that there is a left neighborhood $N_\varepsilon(y^o)$ in which $u(g) + \delta v(g) - g$ is constant if $y^o > [u']^{-1}(1 - \delta(1-d)/n)$ (since otherwise $y(g)$ would be constant). Moreover, since y^o is a stable steady state and $y(g)$ is strictly increasing, $g \in N_{\varepsilon'}(y^o)$ implies $y(g) \in N_{\varepsilon'}(y^o)$ for any open left neighborhood $N_{\varepsilon'}(y^o) = (y^o - \varepsilon', y^o) \subset N_\varepsilon(y^o)$. These observations imply:

Lemma A.7. *If $y^o > [u']^{-1}(1 - \delta(1-d)/n)$, then there is a left neighborhood $N_\varepsilon(y^o)$ in which*

$$y'(g) = \frac{n}{n-1} \left(\frac{1-u'(g)}{\delta} - \frac{1-d}{n} \right) \quad (10)$$

Proof. There is a $N_\varepsilon(y^o)$ and a constant K such that $\delta v(g) = K + g - u(g)$ for $g \in N_\varepsilon(y^o)$. Hence $v(g)$ is differentiable in $N_\varepsilon(y^o)$. Moreover, $y(g) \in N_\varepsilon(y^o)$ for all $g \in N_\varepsilon(y^o)$. Hence $u(y(g)) + \delta v(y(g)) - y(g)$ is constant in $g \in N_\varepsilon(y^o)$ as well. These observations and the definition of $v(g)$ imply that $v'(g) = \frac{1-d}{n} + (1 - \frac{1}{n})y'(g)$ in $N_\varepsilon(y^o)$. Given that $u'(g) + \delta v'(g) = 1$ in $g \in N_\varepsilon(y^o)$, we must have: $u'(g) + \delta v'(g) = u'(g) + \delta [\frac{1-d}{n} + (1 - \frac{1}{n})y'(g)] = 1$ which implies (10) for any $g \in N_\varepsilon(y^o)$. ■

Let g^m be a sequence in $N_\varepsilon(y^o)$ such that $g^m \rightarrow y^o$. We must have

$$\begin{aligned} y^-(y^o) &= \lim_{\Delta \rightarrow 0} \frac{y(y^o) - y(y^o - \Delta)}{\Delta} = \\ &= \lim_{\Delta \rightarrow 0} \lim_{m \rightarrow \infty} \frac{y(g^m) - y(g^m - \Delta)}{\Delta} = \\ &= \lim_{m \rightarrow \infty} \lim_{\Delta \rightarrow 0} \frac{y(g^m) - y(g^m - \Delta)}{\Delta} = \frac{n}{n-1} \left(\frac{1-u'(y^o)}{\delta} - \frac{1-d}{n} \right) \end{aligned} \quad (11)$$

where $y^-(y^o)$ is the left derivative of $y(g)$ at y^o , the second equality follows from continuity and the last equality follows from Lemma A.7 and the fact that under the starting assumption we have $y^o > [u']^{-1}(1 - \delta(1-d)/n) > [u']^{-1}(1 - \delta(1-d)/n)$. Consider a state $(y^o - \Delta)$. For y^o to be stable we need that for any small Δ :

$$y(y^o - \Delta) \geq y^o - \Delta = y(y^o) + (y^o - \Delta) - y^o$$

where the equality follows from the fact that $y(y^o) = y^o$. As $\Delta \rightarrow 0$, this implies $y^-(y^o) \leq 1$ in $N_\varepsilon(y^o)$. By (11), we must therefore have: $\frac{n}{n-1} \left(\frac{1-u'(y^o)}{\delta} - \frac{1-d}{n} \right) \leq 1$. This implies: $y^o \leq [u']^{-1}(1 - \delta(1-d)/n)$, a contradiction.

Case 2. Assume now that $y^o = W/d$ and consider first the case in which it is a strict local maximum of the objective function $u(y) + \delta v(y) - y$. In this case in a left neighborhood $N_\varepsilon(y^o)$, we have that the upper bound $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n}y(g)$ is binding: implying $y(g) = W + (1-d)g$

in $N_\varepsilon(y^\circ)$. We must therefore have a sequence of points $g^m \rightarrow y^\circ$ such that $g^m = y(g^{m-1})$ and $y(g^m) = W + (1-d)g^m \forall m$. Given this, we can write:

$$\begin{aligned} v(g^m) &= u(g^{m+1}) + \delta v(g^{m+1}) = u(g^{m+1}) + \delta [u(g^{m+2}) + \delta v(g^{m+2})] \\ &= \sum_{j=0}^{\infty} \delta^j u(W + (1-d)g^{m+j}) \end{aligned}$$

We therefore must have that $v(g^m)$ is differentiable and $\delta v'(g^m) = \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j})$. Since $u'(g^m) + \delta v'(g^m) \geq 1$, we have $u'(g^m) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j}) \geq 1$ for all m . Consider the limit as $m \rightarrow \infty$. Since $u'(g)$ is continuous and $g^m \rightarrow y^\circ$, we have:

$$\begin{aligned} 1 &\leq \lim_{m \rightarrow \infty} \left[u'(g^m) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(W + (1-d)g^{m+j}) \right] \\ &= u'(y^\circ) + \sum_{j=0}^{\infty} [\delta(1-d)]^{j+1} u'(y^\circ) = \frac{u'(y^\circ)}{1 - \delta(1-d)} \end{aligned}$$

This implies $y^\circ \leq [u']^{-1}(1 - \delta(1-d)) < [u']^{-1}(1 - \delta(1-d/n))$, a contradiction. Assume now that $y^\circ = W/d$, but it is not a strict maximum of $u(y) + \delta v(y) - y$ in any left neighborhood. It must be that $u(y) + \delta v(y) - y$ is constant in some left neighborhood $N_\varepsilon(y^\circ)$. If this were not the case, then in any left neighborhood we would have an interval in which $y(g)$ is constant, but this is impossible by Lemma A.6. But then if $u(y) + \delta v(y) - y$ is constant in some $N_\varepsilon(y^\circ)$, the same argument as in Step 1 implies a contradiction. ■

3 Proof of Proposition 4

Proposition 4. For any $d > 0$ and n , there is a $\bar{\delta} < 1$ such that the most efficient SPE path in a RIE and the most efficient SPE path in a IIE coincide with the Pareto efficient investment path for any $\delta > \bar{\delta}$. Hence, neither the most efficient SPE path in a RIE nor the most efficient SPE path in a IIE are characterized by gradualism for any $\delta > \bar{\delta}$.

We first show that there is a $\delta_1 < 1$, such that for $\delta > \delta_1$ the efficient path is a SPE path in a irreversible investment economy. To this goal, we first define the equilibrium strategies and establish some key properties. Let $y^M(g; d, \delta)$, $v^M(g; d, \delta)$ be, respectively, the investment function and the value function of the Markov equilibrium with the lowest steady state characterized in Proposition 2 when the discount factor is δ and the rate of depreciation is d . Let $g^M(d, \delta) = [u']^{-1}(1 - \delta(1-d)/n)$ be the associated steady state. It is easy to see that, for any d and n , $g^M(d, \delta) < y_P^*(\delta, d, n)$ for all $\delta \in [0, 1]$. Define $y_j^M(g; d, \delta)$ recursively with $y_0^M(g; d, \delta) = g$ and $y_j^M(g; d, \delta) = y^M(y_{j-1}^M(g; d, \delta); d, \delta)$. For any g , $y_j^M(g; d, \delta) \rightarrow g^M(d, \delta)$ as $j \rightarrow \infty$. It follows that $\lim_{\delta \rightarrow 1} [(1-\delta)v^M(g; d, \delta)] = (W - dg^M(d, 1))/n + u(g^M(d, 1))$. Let $y^P(g; d, \delta)$ be the

efficient investment function characterized in Section 3 with steady state $g^P(d, \delta) = y_P^*(\delta, d, n)$, and let $v^P(g; d, \delta)$ be the associated expected utility for a player. Similarly, it is easy to see that $\lim_{\delta \rightarrow 1} [(1 - \delta) v^P(g; d, \delta)] = (W - dg^P(d, 1)) / n + u(g^P(d, 1))$, where $y^P(g; d, \delta)$ be the efficient investment function characterized in Section 3 with steady state $g^P(d, \delta) = y_P^*(\delta, d, n)$. It follows that $\lim_{\delta \rightarrow 1} [(1 - \delta) v^P(g; d, \delta)] > \lim_{\delta \rightarrow 1} [(1 - \delta) v^M(g; d, \delta)]$.

Associated to an aggregate investment function $y^l(g; d, \delta)$, $l = \{M, P\}$, we have the individual contribution function: $i^l(g; d, \delta) = [y^l(g; d, \delta) - (1 - d)g] / n$. To construct the equilibrium, consider the following trigger strategies. If $g_\tau = y_\tau^P(g_0; d, \delta)$ for all $\tau \leq t$, then $i^t(g_t; d, \delta) = i^P(g; d, \delta)$, where $i_j^t(g_t)$ is the investment at time t of an agent. If $\exists \tau \leq t$ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$, then $i^t(g_t) = i^M(g; d, \delta)$. Note that, by construction, deviations are not profitable after a τ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1 - \delta) v^M(g; d, \delta) < (1 - \delta) v^P(g; d, \delta)$, so there must be a $\delta_1 < 1$, such that for $\delta > \delta_1$ no deviation is profitable.

The result that we also have a $\delta_2 < 1$, such that for $\delta > \delta_2$ the efficient path is a SPE path in a reversible investment economy can be proven analogously. From Battaglini et al. [2012], we know that there is a Markov equilibrium $\tilde{y}^M(g; d, \delta)$, $\tilde{v}^M(g; d, \delta)$ with steady state $\tilde{g}^M(d, \delta) \leq [u']^{-1} (1 - \delta(1 - d)/n)$, and so strictly lower than the steady state $g^P(d, 1)$ of the planner's solution for all $\delta \in [0, 1]$. Proceeding exactly as above we can see that $\lim_{\delta \rightarrow 1} [(1 - \delta) v^P(g; d, \delta)] > \lim_{\delta \rightarrow 1} [(1 - \delta) \tilde{v}^M(g; d, \delta)]$. Associated to an aggregate investment function $\tilde{y}^M(g; d, \delta)$ we define as above the individual contribution function: $\tilde{i}^M(g; d, \delta) = [\tilde{y}^M(g; d, \delta) - (1 - d)g] / n$. To construct the equilibrium, consider the following trigger strategies. If $g_\tau = y_\tau^P(g_0; d, \delta)$ for all $\tau \leq t$, then $i^t(g_t; d, \delta) = i^P(g; d, \delta)$, where $i^t(g_t)$ is the investment at time t of an agent. If $\exists \tau \leq t$ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$, then $i^t(g_t) = \tilde{i}^M(g; d, \delta)$. Note that, by construction, deviations are not profitable after a τ such that $g_\tau \neq y_\tau^P(g_0; d, \delta)$. For the remaining histories note that the average utility of a deviating agent must converge to $(1 - \delta) \tilde{v}^M(g; d, \delta) < (1 - \delta) v^P(g; d, \delta)$, so there must be a $\delta_2 < 1$, such that for $\delta > \delta_2$ no deviation is profitable. Given this, the statement of the proposition follows immediately by defining $\bar{\delta} = \max(\delta_1, \delta_2)$. ■